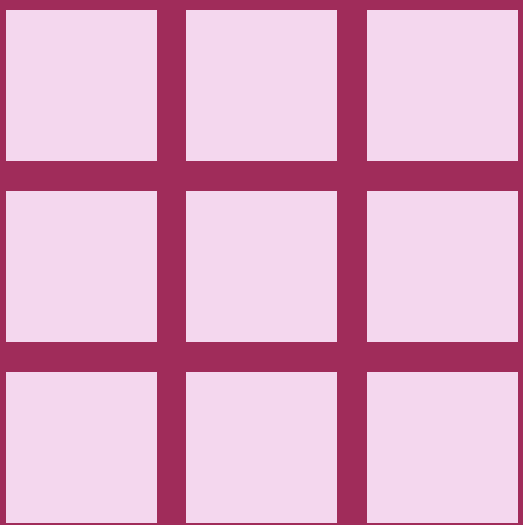
N. V. Kopchenova and I.A. Maron

COMPUTATIONAL MATHEMATICS

Worked Examples and Problems with Elements of Theory



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Н. В. Копченова, И. А. Марон

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N. V. Kopchenova and I. A. Maron

Computational Mathematics

WORKED EXAMPLES AND PROBLEMS WITH ELEMENTS OF THEORY

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PREFACE

Reasonable utilization of modern computer techniques is out of the question without skillful application of approximate and numerical analysis, which explains the remarkable growth of interest in the methods of approximate calculation.

Computational mathematics as a teaching discipline has an increasing place in the syllabuses of engineering and economics faculties and colleges and with it the demand for textbooks in this discipline has grown. A particularly acute need for manuals on solving problems in computational mathematics is felt.

The present book is an attempt to provide such a manual.

The book has the following structure: each section begins with a brief theoretical introduction on setting the problem, giving working formulas, computational schemes, estimates of error, and a comparison of different methods from the angle of their laboriousness, their attainable degree of accuracy, and the convenience of solving them on a computer. There follows a detailed solution of model problems illustrating the appropriate algorithms. At the end of the section problems for independent practice are suggested. Answers are given for most of these problems.

For better understanding of the essence of the matter, most of the problems for which solutions are provided have been selected so that the computations will not be especially cumbersome. It is suggested that desk computers should be used in the initial stage of instruction, and for group study.

The book is intended primarily for engineering students but may also prove useful for economics students, for graduate engineers, and for post graduate students and scientific workers in the applied sciences. The material, we would emphasize, does not go beyond the limits of the syllabus in computational mathematics.

The authors take this opportunity to express their deep gratitude to Kh.L. Smolitsky and I.M. Stesin for reading the manuscript. Their advice and remarks helped to improve the book. We also sincerely thank L.Z. Rumshisky for editing the whole work and for his part in the writing of chapters one and six. We

are aware that the work as it stands is not free of error, the more so that it is the first teaching manual of this kind published in our country. We shall therefore, be grateful for the advice and criticism of readers, which should be addressed to 117071, Moscow V-71, Leninsky Prospect, 15, "Nauka" Publishers Main Editorial Office for Physico-Mathematical Literature.

The Authors

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1

APPROXIMATE COMPUTATIONS AND ERROR ESTIMATION IN PERFORMING COMPUTATIONS

1.1 APPROXIMATE NUMBERS, THEIR ABSOLUTE AND RELATIVE ERRORS

In performing computations, as a rule, we deal with approximate values of quantities, i.e. with *approximate numbers*. The initial data themselves are usually given with some errors, which are then redoubled in the course of calculations by rounding errors, the use of approximate formulas, and so on. A reasonable error estimation enables us to decide on the number of digits to be retained in intermediate operations, as well as in the final result.

The error of an approximate number a, i.e. the difference $a-a_0$ between it and the exact value a_0 is usually not known.

To estimate the error of an approximate number a means to establish an inequality of the form

$$|a-a_0| \leqslant \Delta_a. \tag{1}$$

The number Δ_a is called *the absolute error of an approximate number* a (sometimes the term the limiting absolute error is used).

The limiting absolute error of an approximate number is any number not less than the absolute error of that number. Hence, it follows logically that any number exceeding the limiting absolute error of a given approximate number can also be called the limiting absolute error of this number. For practical purposes it is convenient to take for Δ_a the smallest number (under the given circumstances) that satisfies inequality (1).

When writing absolute errors it is common to give two or three significant digits (when counting significant digits the zeros on the left should be neglected; for instance, the number 0.010030 has five significant digits). In an approximate number a one should not retain the digits which are to be rounded in its absolute error Δ_a.

Example 1. The length and width of a room measured accurate to 1 cm are: $a=5.43$ m and $b=3.82$ m. Estimate the error in determining the area of the room $S=ab=20.7426$ m^2.

Solution. By hypothesis, $\Delta_a=0.01$ m, $\Delta_b=0.01$ m. The extreme possible values of the area are equal to

$$(a+0.01)(b+0.01)=20.8352 \text{ m}^2,$$
$$(a-0.01)(b-0.01)=20.6502 \text{ m}^2;$$

comparing them with the above calculated value of S, we get the following estimate

$$|S-S_0|\leqslant 0.0926,$$

which enables us to indicate the absolute error of the number S in the form $\Delta_S=0.0926$ m^2.

Here it is reasonable to round the value of Δ_S, for instance, in the following way: $\Delta_S=0.093$ m^2 or $\Delta_S=0.10$ m^2 (absolute errors are usually rounded off to a larger number!). In this case the approximate value of the area can be written as $S=20.743$ m^2, or $S=20.74$ m^2, or even $S=20.7$ m^2.

Example 2. A computer is designed to receive numbers only with three significant digits. With what accuracy can the numbers π and $\frac{1}{3}$ be introduced into it?

Solution. Suppose $\pi\approx 3.14=a$ instead of $\pi=3.141592\ldots$, the error of the number a can be estimated by the number $\Delta_a=0.0016$. Suppose $\frac{1}{3}\approx 0.333=b$; the error of the number b can be estimated by the number $\Delta_b=0.00034$ or $\Delta_b=0.0004$.

The relative error δ_a of an approximate number a is the ratio of the absolute error Δ_a of the number to the modulus (absolute value) of the number a, i.e.

$$\delta_a=\frac{\Delta_a}{|a|} \quad (a\neq 0). \tag{2}$$

The relative error is usually expressed in percentage, and is given as a two- or three-digit number.

Sometimes the relative error is understood as the ratio $\frac{\Delta_a}{|a_0|}$, where a_0 is the corresponding exact (but unknown!) value of the number; if the relative error of the number a does not exceed 5 per cent, then the difference between the ratios $\frac{\Delta_a}{|a|}$ and $\frac{\Delta_a}{|a_0|}$ affects only the second digit of the error which is insignificant.

Example 3. Determine the relative error of the number S in Example 1.

Solution. $S = 20.7426$, $\Delta_S = 0.0926$, therefore

$$\delta_S = \frac{0.0926}{20.7426} = \frac{926}{207\,426} = 0.0045 = 0.45\%.$$

In many applied sciences and technology the accuracy of approximate numbers is usually characterized by their relative error.

The relative error of an approximate number is associated with the number of correct digits. *Correct digits* are counted from the first significant digit of a number to the first significant digit of its absolute error: for instance, the number $S = 20.7426$ with the absolute error $\Delta_S = 0.0926$ has three correct digits (2, 0, 7), the rest of the digits being doubtful.

We may assume roughly that the presence of only one correct digit corresponds to a relative error of the order of 10 per cent, of two correct digits to an error of the order of 1 per cent, of three correct digits to an error of the order of 0.1 per cent, and so forth.

In mathematical tables all the numbers are rounded to correct digits, i.e. all indicated significant digits are correct, and the absolute error does not exceed half the unit of the last digit retained. For instance, if a table gives $e = 2.718$, then the absolute error does not exceed $0.5 \cdot 10^{-3}$.

In final results of calculations, besides all correct digits only one doubtful digit is retained.

In intermediate results we usually retain two or three doubtful digits in order not to accumulate extra errors due to rounding.

Example 4. Round off the number $S = 20.7426$ given in Example 1 to correct digits.

Solution. Since the number S has three correct digits, it can be written as

$$S = 20.7.$$

However, in this case the quantity 0.0426 discarded in rounding off must be added to the absolute error $\Delta_S = 0.0926$. The new absolute error $\Delta_S = 0.136$ thus obtained makes us consider as doubtful already the third digit of the number S, and, consequently, we have to round off the number S to *two* digits:

$$S = 21 \quad (\Delta_S = 0.44 < 0.5).$$

This example illustrates that rounding off of results of computation to correct digits is not always advisable.

Note. According to standard practice, we use here *the following rounding-off rule*: if the first discarded digit exceeds or equals 5, add 1 to the last retained digit.

Problems

1. Rounding the following numbers to three significant digits, determine the absolute Δ and the relative δ errors, of the obtained approximate numbers:

(a) 2.1514, (b) 0.16152, (c) 0.01204, (d) 1.225, (e) —0.0015281, (f) —392.85, (g) 0.1545, (h) 0.003922, (i) 625.55, (j) 94.525.

2. Determine the absolute error of the following approximate numbers given their relative errors:

(a) $a=13\,267$, $\delta=0.1\%$, (b) $a=2.32$, $\delta=0.7\%$, (c) $a=35.72$, $\delta=1\%$, (d) $a=0.896$, $\delta=10\%$, (e) $a=232.44$, $\delta=1\%$.

3. In measuring some angles the following numbers were obtained: $\alpha_1=21°37'3''$, $\alpha_2=45°$, $\alpha_3=1°10''$, $\alpha_4=75°20'44''$.

Determine the relative errors of the numbers α_1, α_2, α_3, α_4, assuming the absolute error of the measurements to be equal to 1".

4. Determine the number of correct digits in the number x given its absolute error:

(a) $x=0.3941$, $\Delta_x=0.25\cdot10^{-2}$, (b) $x=0.1132$, $\Delta_x=0.1\cdot10^{-3}$, (c) $x=38.2543$, $\Delta_x=0.27\cdot10^{-2}$, (d) $x=293{,}481$, $\Delta_x=0.1$, (e) $x=2.325$, $\Delta_x=0.1\cdot10^{-1}$, (f) $x=14.00231$, $\Delta_x=0.1\cdot10^{-3}$, (g) $x=0.0842$, $\Delta_x=0.15\cdot10^{-2}$, (h) $x=0.00381$, $\Delta_x=0.1\cdot10^{-4}$, (i) $x=-32.285$, $\Delta_x=0.2\cdot10^{-2}$, (j) $x=-0.2113$, $\Delta_x=0.5\cdot10^{-2}$.

5. Determine the number of correct digits in the number a given its relative error:

(a) $a=1.8921$, $\delta_a=0.1\cdot10^{-2}$, (b) $a=0.2218$, $\delta_a=0.2\cdot10^{-1}$, (c) $a=22.351$, $\delta_a=0.1$, (d) $a=0.02425$, $\delta_a=0.5\cdot10^{-2}$, (e) $a=0.000135$, $\delta_a=0.15$, (f) $a=9.3598$, $\delta_a=0.1\%$, (g) $a=0.11452$, $\delta_a=10\%$, (h) $a=48361$, $\delta_a=1\%$, (i) $a=592.8$, $\delta_a=2\%$, (j) $a=14.9360$, $\delta_a=1\%$.

1.2 ADDING AND SUBTRACTING APPROXIMATE NUMBERS

1. *The absolute error of an algebraic sum of several approximate numbers is equal to the sum of the absolute errors of the numbers:* if

$$S=a_1+a_2+\ldots+a_n,$$

then

$$\Delta_S=\Delta_{a_1}+\Delta_{a_2}+\ldots+\Delta_{a_n}. \tag{1}$$

With a large number of terms the absolute error of the sum as estimated by formula (1) turns out to be too high, since a partial compensation of errors of opposite signs usually takes place.

If among the terms there is a number, whose absolute error exceeds essentially the absolute errors of the rest of the terms, then the absolute error of the sum is considered to be equal to this greatest error. In this case it is advisable to retain as many

decimal digits as are contained in the term with the greatest absolute error.

Let us show how approximate numbers are added and errors are estimated.

Example 1. Find the sum of the approximate numbers 0.348; 0.1834; 345.4; 235.2; 11.75; 9.27; 0.0849; 0.0214; 0.000354, assuming all their digits being correct, i.e. assuming that the absolute error of each term does not exceed half the unit of the junior digit retained.

Solution. Two numbers 345.4 and 235.2 have the maximum absolute error $\Delta = 0.05$. Therefore we may consider that the absolute error of the sum amounts to $2\Delta = 0.10$. Since the terms are not numerous, we retain only one extra digit, i.e. we round off the terms to 0.01:

$$\begin{array}{r} 345.4 \\ 235.2 \\ 11.75 \\ 9.27 \\ 0.35 \\ 0.18 \\ 0.08 \\ 0.02 \\ 0.00 \\ \hline S = 602.25. \end{array}$$

In the final result the extra digit is discarded:

$$S = 602.2.$$

We add the rounding-off error 0.05 to the above mentioned absolute error 0.10, which yields

$$\Delta_S = 0.15 \text{ or } \Delta_S = 0.2.$$

It is worth noting, that in this example taking into account all the errors of the terms according to formula (1) would make the computation more cumbersome without gaining a substantially more accurate result.

2. *If the terms have one and the same sign, the relative error* δ_S *of their sum does not exceed the maximum relative error of any of the terms:*

$$\min \delta_{a_k} \leqslant \delta_S \leqslant \max \delta_{a_k} \ (a_k > 0, \ k = 1, \ 2, \ \ldots, \ n). \qquad (2)$$

Example 2. Estimate the relative error of the sum of numbers in Example 1 and compare it with the relative errors of the terms.

Solution. The relative error of the sum S is

$$\delta_S = \frac{0.10}{602.25} = 0.017\%,$$

those of the summands being

$$\frac{0.5}{348}=0.15\%;\quad \frac{0.5}{1834}=0.027\%;\quad \frac{0.5}{3454}=0.015\%;$$
$$\frac{0.5}{2352}=0.022\%;\quad \frac{0.5}{1175}=0.043\%;\quad \frac{0.5}{927}=0.054\%;$$
$$\frac{0.5}{849}=0.059\%;\quad \frac{0.5}{214}=0.24\%;\quad \frac{0.5}{354}=0.15\%.$$

We see that the bulk of the sum is contributed by the terms 345.4 and 235.2 with the relative errors 0.015% and 0.022%, respectively, and that the relative error of the sum is contained just between these numbers.

3. *The relative error of a difference of two positive numbers is more than the relative errors of these numbers*, especially if they are nearly equal (i.e. if their difference is small as compared with the numbers themselves). This amounts to *a loss of accuracy when subtracting nearly equal numbers*, which should be taken into account when choosing a computational scheme.

Example 3. Given numbers $a=1.137$ and $b=1.073$ with the absolute errors $\Delta_a=\Delta_b=0.011$. Estimate the error of their difference $c=a-b$.

Solution.

$$c=0.064,\quad \Delta_c=\Delta_a+\Delta_b=0.022,\quad \delta_c=\frac{22}{64}=35\%.$$

Thus, the final result has no correct digit, though the numbers themselves have the relative errors

$$\delta_a\approx\delta_b\approx 1\%.$$

Example 4. Compute $1-\cos 1^\circ$ with the aid of four-digit tables of trigonometric functions and estimate the error of the result.

Solution. Taking $\cos 1^\circ$ directly from the tables, we get

$$a=\cos 1^\circ=0.9998,\ \Delta_a=0.00005$$

and

$$b=1-\cos 1^\circ=0.0002,\ \Delta_b=0.00005,$$

which gives the relative error $\delta_b=\frac{0.5}{2}=25\%$.

Let us transform the formula for computing b:

$$b=1-\cos 1^\circ=2\sin^2 0^\circ 30'.$$

From the same tables we find

$$c=\sin 0^\circ 30'=0.0087,\quad \Delta_c=0.00005,\quad \delta_c=\frac{0.5}{87}=0.58\%,$$

$$b=2c^2=0.000151,\quad \delta_b=\delta_{c^2}=2\delta_c=1.16\%\ \text{(see Sec. 1.3)},$$

and consequently

$$\Delta_b = b\delta_b = 0.000151 \cdot 0.0116 = 0.0000018.$$

Thus, this transformation has enabled us to compute the required number b with two correct digits and reduce its relative error by more than 20 times!

Problems

1. Find the sums of approximate numbers and indicate their errors: (a) $0.145 + 321 + 78.2$ (all correct digits), (b) $0.301 + 193.1 + 11.58$ (all correct digits), (c) $398.5 - 72.28 + 0.34567$ (all correct digits), (d) $x_1 + x_2 - x_3$, where $x_1 = 197.6$, $\Delta_{x_1} = 0.2$, $x_2 = 23.44$, $\Delta_{x_2} = 0.22$, $x_3 = 201.55$, $\Delta_{x_3} = 0.17$.

1.3 MULTIPLYING AND DIVIDING APPROXIMATE NUMBERS

1. *When multiplying and dividing approximate numbers their relative* (not absolute!) *errors are added; the relative error of the expression*

$$r = \frac{a_1 a_2 \ldots a_m}{b_1 b_2 \ldots b_n} \tag{1}$$

is estimated by the quantity

$$\delta_r = \delta_{a_1} + \delta_{a_2} + \ldots + \delta_{a_m} + \delta_{b_1} + \delta_{b_2} + \ldots + \delta_{b_n}. \tag{2}$$

If the number $m + n$ is rather large, then it is more expedient to use *the statistic estimate* which takes into account a partial compensation of errors of different signs: if all the numbers a_i, b_j $(i = 1, 2, \ldots, m;\ j = 1, 2, \ldots, n)$ have roughly the same relative error δ, then the relative error of expression (1) is taken to be equal

$$\delta_r = \sqrt{3(n+m)}\,\delta \quad (n + m > 10). \tag{3}$$

If the relative error of one of the numbers a_i, b_j exceeds considerably the relative errors of the remaining numbers, then the relative error of expression (1) is considered to be equal to this greatest error. It is good practice to retain as many significant digits as are contained in the number with the greatest relative error.

2. *The absolute error of expression* (1) *is computed by its relative error:*

$$\Delta_r = |r|\,\delta_r.$$

Let us illustrat how approximate numbers are multiplied and divided, and the error of the final result is estimated.

Example 1. Compute the expression

$$r = \frac{3.2 \cdot 356.7 \cdot 0.04811}{7.1948 \cdot 34.56},$$

assuming that all the numbers are given with correct digits, i.e. that their absolute errors do not exceed half the unit of the junior digit retained.

Solution. The greatest relative error is possessed by the number $a = 3.2$, which contains only two correct digits (as compared with four or five correct digits contained by the rest of the numbers):

$$\delta_a = \frac{0.5}{32} = 1.6\%.$$

Therefore, we can consider the relative error of the final result to be equal to $\delta_r = 1.6\%$, i.e. it contains not more than two correct digits. Since the quantity of the given numbers is rather small, we retain only one extra digit in calculations, rounding off all the numbers to three digits:

$$r = \frac{3.2 \cdot 357 \cdot 0.0481}{7.19 \cdot 34.6} = 0.221$$

(this result can be obtained by means of a slide rule, since a higher accuracy is not required!).

The absolute error of the result is determined by its relative error and the numerical value found:

$$\Delta_r = r\delta_r = 0.221 \cdot 0.016 = 0.0036.$$

Rounding off the result to correct digits, we discard the extra digit and get

$$r = 0.22$$

with the absolute error $\Delta_r < 0.005$.

Problems

1. Find the products of approximate numbers and give their errors (assuming that all digits of the given numbers are correct):
(a) $3.49 \cdot 8.6$, (b) $25.1 \cdot 1.743$, (c) $0.02 \cdot 16.5$, (d) $0.253 \cdot 654 \cdot 83.6$, (e) $1.78 \cdot 9.1 \cdot 1.183$, (f) $482.56 \cdot 7256 \cdot 0.0052$.

2. Find the quotients of approximate numbers:
(a) $5.684 : 5.032$, (b) $0.144 : 1.2$, (c) $216 : 4$, (d) $726.676 : 829$, (e) $754.9367 : 36.5$, (f) $7.3 : 4491$.

3. The sides of a rectangle are equal to (4.02 ± 0.01) m, (4.96 ± 0.01) m. Compute the area of the rectangle.

4. The sides containing the right angle in a right-angled triangle are equal to (12.10 ± 0.01) cm, (25.21 ± 0.01) cm. Compute the tangent of the angle opposite the first side.

5. Measuring the radius R of a circle accurate to 0.5 cm yields the result equal to 12 cm. Find the absolute and relative errors in measuring the area of the circle.

6. Each edge of a cube measured to within 0.02 cm turned out to be equal to 8 cm. Find the absolute and relative errors in measuring the volume of the cube.

7. The altitude h and the radius R of the base circle of the cylinder are measured accurate to 0.5%. What is the limiting relative error in computing the volume of the cylinder?

1.4 ERRORS IN COMPUTING THE VALUES OF A FUNCTION

1. Functions of one variable. The absolute error of a differentiable function $y = f(x)$ due to a sufficiently small error of the argument Δ_x is estimated by the quantity

$$\Delta_y = |f'(x)| \Delta_x \tag{1}$$

for $f'(x) \neq 0$.

If the values of the function $f(x)$ are positive, then the relative error is estimated as follows

$$\delta_y = \frac{|f'(x)|}{|f(x)|} \Delta_x = |[\ln f(x)]'| \Delta_x. \tag{2}$$

In particular, for the basic elementary functions we obtain the following rules.

(a) *Power function* $y = x^a$. The absolute error of the power function is

$$\Delta_y = a x^{a-1} \Delta_x. \tag{3}$$

The relative error of the power function is

$$\delta_y = |a| \delta_x. \tag{4}$$

For example, the relative error of the square x^2 is twice the relative error of the base x, the relative error of the square root $\sqrt{x}$ is half the relative error of the radicand x, the relative error of the reciprocal $\frac{1}{x}$ is equal to that of the number x itself.

(b) *Exponential function* $y = a^x$ $(a > 0)$. The absolute error of the exponential function is

$$\Delta_y = a^x \ln a \cdot \Delta_x. \tag{5}$$

The relative error of the exponential function is

$$\delta_y = \Delta_x \ln a. \tag{6}$$

Here the relative error of the function is proportional to the absolute error of the argument.

Hence, for the function $y=e^x$ we get

$$\delta_y=\Delta_x. \tag{7}$$

(c) *Logarithmic function* $y=\ln x$. The absolute error of the natural logarithm of a number is equal to the relative error of the number itself:

$$\Delta_y=\frac{1}{x}\Delta_x=\delta_x. \tag{8}$$

For the common logarithm $y=\log x$ we have

$$\Delta_y=0.4343\,\delta_x, \tag{9}$$

whence it follows that, when dealing with numbers having m correct digits, $(m+1)$-digit tables of logarithms should be used.

(d) *Trigonometric functions*. The absolute errors of sine and cosine do not exceed those of the argument:

$$\Delta_{\sin x}=|\cos x|\,\Delta_x\leqslant\Delta_x,\quad \Delta_{\cos x}=|\sin x|\,\Delta_x\leqslant\Delta_x. \tag{10}$$

The absolute errors of tangent and cotangent are always more than those of the argument:

$$\Delta_{\tan x}=(1+\tan^2 x)\,\Delta_x\geqslant\Delta_x,\quad \Delta_{\cot x}=(1+\cot^2 x)\,\Delta_x\geqslant\Delta_x. \tag{11}$$

Example 1. The diameter of a circle measured to within 1 mm amounts to $d=0.842$ m. Compute the area of the circle.

Solution. The area of a circle $S=\frac{\pi d^2}{4}$. Since the number π can be taken with any degree of accuracy, the error of computation of the area is determined by the error of computation of d^2. The relative error of d^2 is

$$\delta_{d^2}=2\delta_d=2\cdot\frac{1}{842}=0.24\%.$$

In order, when rounding off the number π, not to increase the relative error

$$\delta_S=\delta\left(\frac{\pi}{4}\right)+2\delta_d,$$

the number π should be taken at least with four correct digits, even better with five. Then we get

$$S=\frac{3.1416}{4}\cdot 0.842^2\ m^2=0.7854\cdot 0.7090\ m^2=0.5568\ m^2.$$

The absolute error of the result amounts to

$$\Delta_S=S\delta_S=0.557\cdot 0.0024=0.0014.$$

We then round off the result to three digits (discarding the extra digit and using the following rule):

$$S=0.557\ m^2,\quad \Delta_S=0.002.$$

Example 2. The angle $x = 25°20'$ is measured accurate to $1'$. Determine $\sin x$ and its absolute error.

Solution. First compute the absolute error of $\sin x$, using formula (10); for this purpose convert $1'$ into the radian measure: $1' = 0.000291$ radian, and compute

$$\Delta_{\sin x} = \cos x \cdot \Delta_x = \cos 25°20' \cdot 0.000291 = 0.00026.$$

Therefore, for computing $\sin x$ four-place tables of trigonometric functions should be used, which gives

$$\sin x = \sin 25°20' = 0.4279.$$

2. Functions of several variables. The absolute error of a differentiable function $y = f(x_1, x_2, \ldots, x_n)$, caused by sufficiently small errors $\Delta_{x_1}, \Delta_{x_2}, \ldots, \Delta_{x_n}$ of the arguments $x_1, x_2, \ldots, x_n$, is estimated by the quantity

$$\Delta_y = \sum_{i=1}^{n} \left| \frac{\partial f}{\partial x_i} \right| \Delta_{x_i}. \tag{12}$$

If the values of the function are positive, then the relative error is estimated by the formula

$$\delta_y = \sum_{i=1}^{n} \frac{1}{f} \left| \frac{\partial f}{\partial x_i} \right| \Delta_{x_i} = \sum_{i=1}^{n} \left| \frac{\partial \ln f}{\partial x_i} \right| \Delta_{x_i}. \tag{13}$$

Example 3. Evaluate the function $u = xy^2z^3$, if

$$x = 37.1, \quad y = 9.87, \quad z = 6.052,$$

and

$$\Delta_x = 0.3, \quad \Delta_y = 0.11, \quad \Delta_z = 0.016.$$

Solution. Here the relative errors of the arguments are equal to

$$\delta_x = \frac{3}{371} = 0.81\%, \quad \delta_y = \frac{11}{987} = 1.12\%, \quad \delta_z = \frac{16}{6052} = 0.26\%.$$

The relative error of the function is

$$\delta_u = \delta_x + 2\delta_y + 3\delta_z = 3.8\%;$$

therefore the computed value of the function must contain not more than two or three digits:

$$u = 801 \cdot 10^3$$

(here it is incorrect to write 801 000 which would have another meaning!).

In this case the absolute error amounts to

$$\Delta_u = u\delta_u = 801 \cdot 10^3 \cdot 0.038 = 30 \cdot 10^3.$$

Here it is advisable to round off the final result to two digits:

$$u = 8.0 \cdot 10^5, \quad \Delta_u = 0.3 \cdot 10^5.$$

Example 4. Compute the value $z = \ln(10.3 + \sqrt{4.4})$, considering all the digits of the approximate numbers $x = 10.3$ and $y = 4.4$ as correct.

Solution. The number y has the relative error $\delta_y = \frac{0.5}{44} = 1.2\%$, therefore $\sqrt{y}$ has the relative error 0.6% and it should be written with three digits:

$$\sqrt{y} = \sqrt{4.4} = 2.10,$$

the absolute error of this root being equal to

$$\Delta_{\sqrt{y}} = 2.10 \cdot 0.006 = 0.013.$$

The absolute error of the sum

$$x + \sqrt{y} = 10.3 + 2.10 = 12.4$$

is estimated by the quantity $0.05 + 0.013 = 0.063$, its relative error being equal to $\frac{0.63}{124} = 0.5\%$.

Using formula (8), we see that the absolute error of the natural logarithm is of the same value, i. e. $\Delta_z = 0.005$. Therefore

$$z = \ln(10.3 + 2.10) = \ln 12.40 = 2.517.$$

Here the result has three correct digits; rounding-off to correct digits is not advisable, since in this case it would be necessary to write the value of Δ_z taking into account the round-off error:

$$z = 2.52, \quad \Delta_z = 0.008.$$

Problems

1. Angles x are measured with the limiting absolute error Δ_x. Determine the absolute and relative errors of the functions $y = \sin x$, $y = \cos x$, and $y = \tan x$. Using the tables, find the values of the functions, retaining only correct digits in the final result:
(a) $x = 11°20'$, $\Delta_x = 1'$, (b) $x = 48°42'31''$, $\Delta_x = 5''$, (c) $x = 45°$, $\Delta_x = 1'$, (d) $x = 50°10'$, $\Delta_x = 0.05°$, (e) $x = 0.45$, $\Delta_x = 0.5 \cdot 10^{-2}$, (f) $x = 1.115$, $\Delta_x = 0.1 \cdot 10^{-3}$.

2. Compute the values of the following functions for the given values of x, and indicate the absolute and relative errors of the results:
(a) $y = x^3 \sin x$ for $x = \sqrt{2}$, putting $\sqrt{2} \approx 1.414$,
(b) $y = x \ln x$ for $x = \pi$, putting $\pi \approx 3.142$,
(c) $y = e^x \cos x$ for $x = \sqrt{3}$, putting $\sqrt{3} \approx 1.732$.

3. Compute the values of the following functions for the given values of the variables. Indicate the absolute and relative errors of the results, considering all the digits of the initial data as correct:

(a) $u = \ln(x_1 + x_2^2)$, $x_1 = 0.97$, $x_2 = 1.132$,

(b) $u = \dfrac{x_1 + x_2^2}{x_3}$, $x_1 = 3.28$, $x_2 = 0.932$, $x_3 = 1.132$,

(c) $u = x_1 x_2 + x_1 x_3 + x_2 x_3$, $x_1 = 2.104$, $x_2 = 1.935$, $x_3 = 0.845$.

4. Determine the relative error of computing the total surface of a truncated cone if the radii of its base circles R and r, and the generatrix l measured to within 0.01 cm are respectively equal to

$$R = 23.64 \text{ cm}, \quad r = 17.31 \text{ cm}, \quad l = 10.21 \text{ cm}.$$

1.5 DETERMINING THE PERMISSIBLE ERROR OF ARGUMENTS BY MEANS OF THE PERMISSIBLE ERROR OF A FUNCTION

This problem has a single-valued solution only for a function of one variable $y = f(x)$: if this function is differentiable and $f'(x) \neq 0$, then

$$\Delta_x = \frac{1}{|f'(x)|} \Delta_y. \tag{1}$$

For a function of several variables $y = f(x_1, x_2, \ldots, x_n)$ the problem is solved only when some additional limitations are introduced. For instance, if the value of one of the arguments is considerably more difficult to be measured or computed more accurately than those of the remaining arguments, then the error of just this argument should be brought in concord with the required error of the function.

If the values of all the arguments can be equally easily determined to any accuracy, then the *principle of equal effects* is usually applied assuming that in formula (12) (Sec. 1.4) all the terms $\left|\dfrac{\partial f}{\partial x_i}\right| \Delta_{x_i}$ are equal; this yields the formula

$$\Delta_{x_i} = \frac{\Delta_y}{n \left|\dfrac{\partial f}{\partial x_i}\right|} \quad (i = 1, 2, \ldots, n). \tag{2}$$

In practice we often come across the problems of the intermediate type, the above mentioned cases being the extremes. Let us now consider some examples by way of illustration.

Example 1. To what accuracy must an angle x be measured in the first quadrant to get the value of $\sin x$ with five correct digits?

Solution. If it is known that the angle $x > 6°$ so that $\sin x > 0.1$, then it is necessary to determine Δ_x so that the inequality $\Delta_{\sin x} < 0.5 \cdot 10^{-5}$ is fulfilled. To this end, according to formula (10) in Sec. 1.4, it is sufficient to take $\Delta_x < 0.5 \cdot 10^{-5}$, i. e. to measure the angle x to within 1″. If, in addition, it is known that $x > 60°$ and, hence, $\cos x < 0.5$, then it is advisable to make use of formula (1), whence

$$\Delta_x = \frac{1}{\cos x} \Delta_{\sin x} > 2 \cdot 0.5 \cdot 10^{-5} = 10^{-5},$$

i. e. it is sufficient to measure x accurate only to 2″.

But if $x < 6°$, say, $1° < x < 6°$, then $0.01 < \sin x < 0.1$, and to ensure five correct digits in the value of $\sin x$ we have to satisfy the inequality $\Delta_{\sin x} < 0.5 \cdot 10^{-6}$, for which purpose we have to measure the angle x to within 0.1″.

Example 2. To what accuracy must the radius R of the base circle and the altitude H of a cylindrical container be determined so that its capacity could be determined accurate to 1%?

Solution. In the formula $V = \pi R^2 H$ the number π can be taken with any number of correct digits and, hence, its error will not affect the result; therefore we may assume $\delta_V = 2\delta_R + \delta_H$. If we can ensure any accuracy in determining R and H, then the principle of equal effects may be used, whereby the shares of $2\delta_R$ and δ_H are equal, each constituting 0.5%. Thus, according to this principle, the radius should be determined with the relative error equal to 0.25%, and the altitude with the relative error of 0.5%. In every day practice we encounter the contrary, when the radius of a container is determined less accurately than the altitude. For instance, if the radius is determined to an accuracy twice lower than that of the altitude, then we assume $\delta_R = 2\delta_H$, and from the condition

$$2\delta_R + \delta_H = 5\delta_H = 1\%$$

we find

$$\delta_H = 0.2\%, \quad \delta_R = 0.4\%.$$

As far as the number π is concerned, in all considered cases it should be taken with the relative error of the order 0.01% so that this error may be neglected in the final result. This means that we may take $\pi = 3.142$ with the relative error $\frac{0.4}{3142} = 0.013\%$, but not $\pi = 3.14$ with the relative error $\frac{0.16}{314} = 0.051\%$.

Example 3. Find the permissible absolute error of the approximate quantities $x = 15.2$, $y = 57$ for which it is possible to find

the value of the function

$$u = 6x^2 (\log x - \sin 2y)$$

accurate to two decimal places.

Solution. We find

$$u = 6x^2 (\log x - \sin 2y) = 6 (15.2)^2 (\log 15.2 - \sin 114°) = 371.9,$$

$$\frac{\partial u}{\partial x} = 12x (\log x - \sin 2y) + 6x \cdot \log e = 88.54,$$

$$\frac{\partial u}{\partial y} = -12x^2 \cos 2y = 1127.7.$$

By hypothesis $\Delta_u = 0.005$. Then, according to the principle of equal effects, we find by formula (2)

$$\Delta_x = \frac{\Delta_u}{2\left|\frac{\partial u}{\partial x}\right|} = \frac{0.005}{2 \cdot 88.54} = 0.28 \cdot 10^{-4},$$

$$\Delta_y = \frac{\Delta_u}{2\left|\frac{\partial u}{\partial y}\right|} = \frac{0.005}{2 \cdot 1127.7} = 0.22 \cdot 10^{-5} = 0''.45.$$

Problems

1. To what accuracy should approximate numbers x be taken to find the values of $\sin x$ with the indicated number m of correct digits?

(a) $x = 1°$, $m = 3$, (b) $x = 25°$, $m = 4$, (c) $x = 30.75°$, $m = 3$, (d) $x = 1.05$, $m = 2$, (e) $x = 0.075$, $m = 2$.

2. To what accuracy are the angles x determined by the values of $\sin x$ taken from a five-place table of trigonometric functions?

(a) $x = 2°1'$, (b) $x = 15°30'$, (c) $x = 44°$, (d) $x = 50°18'$, (e) $x = 65°23'$, (f) $x = 87°$.

3. To what accuracy can we determine the number x by its logarithm with the aid of a five-place table of logarithms if the number is found within the indicated range?

(a) $300 < x < 400$, (b) $35 < x < 40$, (c) $1.5 < x < 1.7$, (d) $3.25 < x < 3.29$, (e) $5000 < x < 6000$.

4. With what number of correct digits should the value of the argument x be taken to obtain the values of the given functions accurate to $0.1 \cdot 10^{-5}$?

(a) $y = x^3 \sin x$, $x = \sqrt{2}$, (b) $y = x \ln x$, $x = \pi$, (c) $y = e^x \cos x$, $x = \sqrt{3}$.

5. With what number of correct digits must the constant term of the equation $x^2 - 2x + \log 2 = 0$ be known to obtain the roots with four correct digits?

6. Find the permissible absolute errors of the arguments, which make it possible to compute the values of the given functions with four correct digits:

(a) $u = \ln(x_1 + x_2^2)$, $x_1 = 0.9731$, $x_2 = 1.13214$,

(b) $u = \frac{x_1 + x_2^2}{x_3}$, $x_1 = 3.2835$, $x_2 = 0.93221$, $x_3 = 1.13214$,

(c) $u = x_1x_2 + x_1x_3 + x_2x_3$, $x_1 = 2.10415$, $x_2 = 1.93521$, $x_3 = 0.84542$.

7. Young's modulus in deflection of a square rod is determined by the formula

$$E = \frac{1}{4}\frac{l^3P}{d^3bs},$$

where l is the length of the bar, b and d are respectively the base and height of the cross section, s is the sagitta, and P is the load. To what accuracy must the length l and sagitta s be measured so that the error of E did not exceed 5.5%, provided P is known to within 0.1%, b and d to within 1%, $l \approx 50$ cm, $s \approx 2.5$ cm?

2

COMPUTING THE VALUES OF FUNCTIONS

When using computing machines to evaluate functions given by formulas, the form in which the formula is written is not at all immaterial. Expressions which are mathematically equivalent are not of the same status as concerns approximate computations. The thing is that the basic operations of most computers are addition, subtraction, multiplication, and division. This gives rise to a necessity of representing a mathematical problem under consideration in the form of a sequence of these elementary operations. Taking into account the restricted storage volume of any machine and the necessity to use the machine time highly economically, it is desirable to split up the operations into repeating cycles and to choose an adequate algorithm. We now consider some typical techniques reducing the computation of some functions to such cycles of elementary operations.

2.1 COMPUTING THE VALUES OF A POLYNOMIAL. HORNER'S SCHEME

Suppose we have a polynomial of degree n,

$$P(x) = a_0 x^n + a_1 x^{n-1} + \ldots + a_n$$

with real coefficients a_k $(k = 0, 1, \ldots, n)$. Suppose it is required to find the value of this polynomial for $x = \xi$:

$$P(\xi) = a_0 \xi^n + a_1 \xi^{n-1} + \ldots + a_n. \qquad (1)$$

The computation of $P(\xi)$ is most conveniently performed as follows. Represent formula (1) as

$$P(\xi) = (\ldots(((a_0 \xi + a_1)\xi + a_2)\xi + a_3)\xi + \ldots + a_n).$$

Introducing the numbers

$$\left.\begin{array}{ll} & b_0 = a_0, \\ c_1 = b_0\xi, & b_1 = a_1 + c_1, \\ c_2 = b_1\xi, & b_2 = a_2 + c_2, \\ \cdots\cdots\cdots & \cdots\cdots\cdots \\ c_n = b_{n-1}\xi, & b_n = a_n + c_n, \end{array}\right\} \qquad (2)$$

we get $b_n = P(\xi)$.

Thus, the computation of the value of the polynomial $P(x)$ for $x = \xi$ is reduced to repeating the following sequence of elementary operations:

$$c_k = b_{k-1}\xi, \quad b_k = a_k + c_k \quad (k = 1, 2, \ldots, n).$$

It is easy to show that the numbers $b_0 = a_0, b_1, \ldots, b_{n-1}$ are coefficients of the polynomial $Q(x)$ obtained as the quotient on dividing the given polynomial $P(x)$ by the binomial $x - \xi$, and $b_n = P(\xi)$ is the remainder.

Thus, formulas (2) enable us, without performing the operation of division, to determine the coefficients of the quotient $Q(x)$, and also the remainder $P(\xi)$. The numbers $b_0, b_1, \ldots, b_n$ are usually found with the aid of the known *Horner's scheme*

$$\begin{array}{r|cccccc} & a_0 & a_1 & a_2 & \ldots & a_n & \xi \\ + & & b_0\xi & b_1\xi & \ldots & b_{n-1}\xi & \\ \hline & b_0 = a_0 & b_1 & b_2 & \ldots & b_n = P(\xi) & \end{array}.$$

The computation of the values of the polynomial $P_n(x)$ according to Horner's scheme requires the performing of n multiplications and $n - k$ additions, where k is the number of coefficients a_i equal to zero. If $a_0 = 1$, then $n - 1$ multiplications are required to be performed. It is proved that for computing polynomials of the general form it is impossible to construct a scheme more economical, as far as the number of operations is concerned, than Horner's scheme.

Example 1. Compute for $x = -1.5$ the value of the polynomial

$$P(x) = x^7 - 2x^6 + x^5 - 3x^4 + 4x^3 - x^2 + 6x - 1.$$

Solution. Using Horner's scheme, we get

$$\begin{array}{r|ccccccccc} & 1 & -2 & 1 & -3 & 4 & -1 & 6 & -1 & -1.5 \\ + & & -1.5 & 5.25 & -9.375 & 18.5625 & -33.8438 & 52.2657 & -87.3985 & \\ \hline & 1 & -3.5 & 6.25 & -12.375 & 22.5625 & -34.8438 & 58.2657 & -88.3985 & = \\ & & & & & & & & & = P(-1.5). \end{array}$$

Thus,

$$P(-1.5) = -88.3985.$$

Problems

1. Given the polynomial

$$P(x) = a_0x^4 + a_1x^3 + a_2x^2 + a_3x + a_4.$$

Find the value of P (3.25) for the coefficients a_0, a_1, a_2, a_3, a_4 given in Table 1.

Table 1

Coefficients a_i $(i = 0, 1, 2, 3, 4)$

	a_0	a_1	a_2	a_3	a_4		a_0	a_1	a_2	a_3	a_4
(a)	7.54	11.08	3.82	0.44	—0.18	(e)	2.79	9.85	14.15	5.38	7.24
(b)	9.36	12.69	14.39	0.79	—0.94	(f)	3.45	—2 91	3.79	—6.75	—2.38
(c)	12.78	14.35	17.19	1.34	—1.72	(g)	4.79	5.38	—2.86	7.31	4.55
(d)	15.65	17.58	21.7	2.78	1.34	(h)	8.34	—7.75	4.53	—9.29	5.79

2. Given the polynomial

$$P(x) = 0.22\ x^5 - 3.27\ x^4 - 2.74\ x^3 + 2.81\ x^2 - 3.36\ x + 2.$$

Find the values of $P(\xi)$, where $\xi = 0.80 + 0.05k$; $k = 0, 1, 2, \ldots, 20$.

2.2 EVALUATING SOME TRANSCENDENTAL FUNCTIONS WITH THE AID OF POWER SERIES

The present section deals only with the transcendental functions which are the sums of their Maclaurin's series

$$f(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!} x^k. \tag{1}$$

Taking the sum of the first several terms of Maclaurin's series we obtain the approximate formula

$$f(x) \approx P_n(x),$$

where

$$P_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}(0)}{k!} x^k.$$

Here the remainder of the series

$$R_n(x) = f(x) - P_n(x)$$

represents the error due to replacing $f(x)$ by the polynomial $P_n(x)$. Estimating the remainder enables us to determine the required number of terms, i. e. the degree n of the polynomial $P_n(x)$.

Note. Since the computation of the total error represents a labour-consuming operation, in practice for ensuring the preassigned degree of accuracy all intermediate computations are performed with one or two extra digits.

1. Computing the values of exponential functions. For an exponential function the following expansion is true

$$e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!} \quad (-\infty < x < \infty). \tag{2}$$

It is convenient to perform computations using the following recurrent notation

$$e^x = \sum_{k=0}^{\infty} u_k, \quad u_k = \frac{x}{k} u_{k-1}, \quad S_k = S_{k-1} + u_k \quad (k = 1, 2, \ldots, n),$$

where $u_0 = 1$, $S_0 = 1$. The number $S_n = \sum_{k=0}^{n} \frac{x^k}{k!}$ is the required approximate result for e^x.

For the remainder of the series the following estimate can be obtained (see [12]):

$$|R_n(x)| < |u_n| \text{ for } 0 < 2|x| \leqslant n;$$

therefore we can terminate the summation process as soon as the last computed term u_k of the series is less by its modulus than the preset permissible error ε:

$$|u_n| < \varepsilon \quad \left(\text{if only } |x| \leqslant \frac{n}{2}\right).$$

With large moduli of x series (2) is hardly fit for computations. In such cases the usual procedure is as follows: represent x in the form of a sum

$$x = E(x) + q,$$

where $E(x)$ is the integral part of x and q its fractional part, $0 \leqslant q < 1$. Then

$$e^x = e^{E(x)} \cdot e^q.$$

The first factor $e^{E(x)}$ is found by means of multiplication

$$e^{E(x)} = \underbrace{e \cdot e \ \ldots \ e}_{E(x) \text{ times}}, \quad \text{if} \quad E(x) > 0,$$

and

$$e^{E(x)} = \underbrace{\frac{1}{e} \cdot \frac{1}{e} \ \ldots \ \frac{1}{e}}_{-E(x) \text{ times}}, \quad \text{if} \quad E(x) < 0.$$

The second factor is computed with the aid of a power expansion

$$e^q = \sum_{n=0}^{\infty} \frac{q^n}{n!}.$$

For $0 \leqslant q < 1$ this series converges rapidly, since

$$0 \leqslant R_n(q) < \frac{1}{n!\,n} q^{n+1}.$$

Example 1. Find $\sqrt{e}$ to within 10^{-5}.

Solution. Use the formula

$$e^{1/2} = \sum_{k=0}^{n} u_k + R_n\left(\frac{1}{2}\right), \tag{3}$$

where $u_0 = 1$, $u_k = \frac{u_{k-1}}{2k}$ $(k = 1, 2, \ldots, n)$. The summands are computed with two extra decimal digits.

We successively have

$$u_0 = 1,$$

$$u_1 = \frac{u_0}{2} = 0.5000000,$$

$$u_2 = \frac{u_1}{4} = 0.1250000,$$

$$u_3 = \frac{u_2}{6} = 0.0208333,$$

$$u_4 = \frac{u_3}{8} = 0.0026042,$$

$$u_5 = \frac{u_4}{10} = 0.0002604,$$

$$u_6 = \frac{u_5}{12} = 0.0000217,$$

$$u_7 = \frac{u_6}{14} = 0.0000016,$$

$$S_7 = \overline{1.6487212.}$$

Rounding the sum to five decimal digits, we get

$$\sqrt{e} = 1.64872.$$

To compute the values of the exponential function a^x $(a > 0)$ use the formula $a^x = e^{x \ln a}$.

2. Computing the sine and cosine values. For computing the values of the functions $\sin x$ and $\cos x$ use is made of the follo-

wing power-series expansions

$$\sin x = \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k+1}}{(2k+1)!} \quad (-\infty < x < \infty), \tag{4}$$

$$\cos x = \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k}}{(2k)!} \quad (-\infty < x < \infty). \tag{5}$$

For large values of x series (4) and (5) converge slowly, but, taking into consideration the periodicity of the functions $\sin x$ and $\cos x$, and reduction formulas for trigonometric functions, it is easy to conclude that it is sufficient to know how to compute $\sin x$ and $\cos x$ for the interval $0 \leqslant x \leqslant \frac{\pi}{4}$. Here we can use the following recurrent formulas:

$$\left.\begin{aligned} &\sin x = \sum_{k=1}^{n} u_k + R_n(x), \\ &u_1 = x, \quad u_{k+1} = -\frac{x^2}{2k(2k+1)} u_k \ (k = 1, 2, \ldots, n-1); \end{aligned}\right\} \tag{6}$$

$$\left.\begin{aligned} &\cos x = \sum_{k=1}^{n} v_n + R_n(x), \\ &v_1 = 1, \quad v_{k+1} = -\frac{x^2}{(2k-1)2k} v_k \ (k = 1, 2, \ldots, n-1). \end{aligned}\right\} \tag{7}$$

Since in the interval $\left(0, \frac{\pi}{4}\right)$ the series (4) is an alternating series with terms monotonically decreasing in modulus, for the remainder term R_n the estimate

$$|R_n| \leqslant \frac{|x|^{2n+1}}{(2n+1)!} = |u_{n+1}|$$

holds true.

Analogously for the series (5): $|R_n| \leqslant |v_{n+1}|$. Hence, the process of computation of $\sin x$ and $\cos x$ may be terminated as soon as we find that $|u_n| \leqslant \varepsilon$, where ε is the specified permissible error.

Example 2. Compute $\sin 23°54'$ to within 10^{-4}.

Solution. We express the argument in radians retaining one extra digit: $x = \text{arc}\, 23°54' = 0.41714$. Using formula (6), we obtain

$$u_1 = x = +0.41714,$$

$$u_2 = -\frac{x^2 u_1}{2 \cdot 3} = -0.01210,$$

$$u_3 = -\frac{x^2 u_2}{4 \cdot 5} = +0.00011,$$

$$u_4 = -\frac{x^2 u_3}{6 \cdot 7} = -0.00000.$$

Hence, rounding off the final result to four decimal digits, we find $\sin 23^\circ 54' = 0.4052$.

Example 3. Compute $\cos 17^\circ 24'$ accurate to 10^{-5}.

Solution. $x = \text{arc } 17^\circ 24' = 0.30369$. Using formula (7), we have

$$v_1 = 1.000000,$$

$$v_2 = -\frac{x^2}{1 \cdot 2} v_1 = -0.046114,$$

$$v_3 = -\frac{x^2}{3 \cdot 4} v_2 = +0.000354,$$

$$v_4 = -\frac{x^2}{5 \cdot 6} v_3 = -0.000001.$$

Hence

$$\cos 17^\circ 24' = 0.95424.$$

3. **Computing the values of the hyperbolic sine and hyperbolic cosine.** Use the power-series expansions

$$\sinh x = \sum_{k=1}^{\infty} \frac{x^{2k-1}}{(2k-1)!} \quad (-\infty < x < \infty), \tag{8}$$

$$\cosh x = \sum_{k=0}^{\infty} \frac{x^{2k}}{(2k)!} \quad (-\infty < x < \infty) \tag{9}$$

and the following recurrent notation

$$\left.\begin{aligned} \sinh x &= \sum_{k=1}^{n} u_k + R_n, \\ u_1 &= x, \quad u_{k+1} = \frac{x^2}{2k(2k+1)} u_k; \end{aligned}\right\} \tag{10}$$

$$\left.\begin{aligned} \cosh x &= \sum_{k=0}^{n} v_k + R_n^*, \\ v_0 &= 1, \quad v_{k+1} = \frac{x^2}{(2k+1)(2k+2)} v_k. \end{aligned}\right\} \tag{11}$$

For $n \geqslant |x| > 0$ we have the estimates (see [12]) $R_n < \frac{1}{3}|u_n|$ and $R_n^* < \frac{2}{3} v_n$.

Example 4. Compute $\sinh 1.4$ accurate to 10^{-6}.

Solution. Using formula (10) we get

$$u_1 = 1.4000000,$$

$$u_2 = \frac{x^2}{2 \cdot 3} u_1 = 0.4573333,$$

$$u_3 = \frac{x^2}{4 \cdot 5} u_2 = 0.0448187,$$

$$u_4 = \frac{x^2}{6 \cdot 7} u_3 = 0.0020915,$$

$$u_5 = \frac{x^2}{8 \cdot 9} u_4 = 0.0000569,$$

$$u_6 = \frac{x^2}{10 \cdot 11} u_5 = 0.0000010.$$

Hence

$$\sinh 1.4 = 1.904301.$$

4. Computing the values of a logarithmic function. Use the expansion in a series in power of $\frac{1-z}{1+z}$:

$$\ln z = -2 \sum_{k=1}^{\infty} \frac{1}{2k-1} \left(\frac{1-z}{1+z} \right)^{2k-1} \quad (0 < z < +\infty).$$

Let x be a positive number. Represent it in the form

$$x = 2^m z,$$

where m is an integer and $1/2 \leqslant z < 1$; then, putting

$$\frac{1-z}{1+z} = \xi,$$

we get

$$\ln x = \ln 2^m z = m \ln 2 + \ln z = m \ln 2 - 2 \sum_{k=1}^{\infty} \frac{1}{2k-1} \xi^{2k-1},$$

where $0 < \xi \leqslant 1/3$. Putting

$$u_k = \frac{\xi^{2k-1}}{2k-1} \quad (k = 1, 2, \ldots, n),$$

we get the following recurrent notation

$$\left.\begin{aligned} &\ln x = m \ln 2 - 2 \sum_{k=1}^{n} u_k + R_n, \\ &u_1 = \xi, \; u_{k+1} = \frac{(2k-1)\xi^2}{2k+1} u_k. \end{aligned}\right\} \tag{12}$$

The summation process is terminated as soon as the inequality $u_n < 4\varepsilon$ is fulfilled, where ε is the permissible error (see [12]).

Example 5. Find $\ln 5$ to within 10^{-6}.

Solution. We shall carry out the computations with two extra digits. Assume $5 = 2^3 \cdot 0.625$. Hence, $z = 0.625$ and $\xi = \frac{1-z}{1+z} = \frac{0.375}{1.625} = 0.23076923$.

Form a column containing the first four summands:

$$u_1 = \xi = 0.23076923,$$
$$u_2 = \frac{\xi^3}{3} = 0.00409650,$$
$$u_3 = \frac{\xi^5}{5} = 0.00013089,$$
$$u_4 = \frac{\xi^7}{7} = 0.00000498,$$
$$\text{the sum} = 0.23500160.$$

By formula (12) we get

$$\ln 5 = 3 \cdot 0.69314718 - 2 \cdot 0.23500160 = 1.609438.$$

Problems

1. Using power-series expansions, form the tables of the values of the following functions with the indicated accuracy ε.

(a) e^x, $x = 0.300 + 0.002k$ $(k = 0, 1, \ldots, 14)$, $\varepsilon = 10^{-5}$,
(b) e^x, $x = 2.500 + 0.002k$ $(k = 0, 1, \ldots, 14)$, $\varepsilon = 10^{-4}$,
(c) e^{-x}, $x = 1.35 + 0.01k$ $(k = 0, 1, \ldots, 14)$, $\varepsilon = 10^{-5}$,
(d) e^{-x}, $x = 0.505 + 0.005k$ $(k = 0, 1, \ldots, 15)$, $\varepsilon = 10^{-5}$,
(e) e^{x^2}, $x = 0.50 + 0.02k$ $(k = 0, 1, \ldots, 15)$, $\varepsilon = 10^{-5}$,
(f) e^{-x^2}, $x = 1.30 + 0.01k$ $(k = 0, 1, \ldots, 15)$, $\varepsilon = 10^{-5}$,
(g) $\frac{1}{2\sqrt{\pi}} e^{-x^2/2}$, $x = 1.78 + 0.03k$ $(k = 0, 1, \ldots, 15)$,
$x = 2.545 + 0.005k$ $(k = 0, 1, \ldots, 15)$.

2. Taking advantage of the expansions of $\sin x$ and $\cos x$ in a power series, compile the tables of the values of the following functions accurate to 10^{-5}.

(a) $\sin x$, $x = 0.345 + 0.005k$ $(k = 0, 1, \ldots, 15)$,
(b) $\sin x$, $x = 1.75 + 0.01k$ $(k = 0, 1, \ldots, 15)$,
(c) $\cos x$, $x = 0.745 + 0.005k$ $(k = 0, 1, \ldots, 15)$,
(d) $\cos x$, $x = 1.75 + 0.01k$ $(k = 0, 1, \ldots, 15)$,
(e) $\frac{\sin x}{x}$, $x = 0.4 + 0.01k$ $(k = 0, 1, \ldots, 15)$,
(f) $\frac{\cos x}{x}$, $x = 0.25 + 0.01k$ $(k = 0, 1. \ldots, 15)$.

3. Taking advantage of the expansions of $\sinh x$ and $\cosh x$ in a power series, compile the tables of the values of the following functions accurate to ε.

(a) $\sinh x$, $x = 0.23 + 0.01k$ $(k = 0, 1, \ldots, 15)$,
$\varepsilon = 10^{-5}$, $x = 2.30 + 0.05k$ $(k = 0, 1, \ldots, 15)$, $\varepsilon = 10^{-4}$,
(b) $\cosh x$ for the same values of x.

2.3 SOME POLYNOMIAL APPROXIMATIONS

Computation with the aid of Taylor's series gives a sufficiently rapid convergence, generally speaking, only for small values of $|x-x_0|$. However, it is often necessary, with the aid of a polynomial of a comparatively low degree, to choose an approximation which would ensure a sufficient accuracy for all points of a given interval. In these cases use is made of expansions of functions obtained by means of Chebyshev's polynomials on a specified interval. Given below are a few examples of such expansions accompanied by certain intervals for them to be used in, as well as by appropriate absolute errors ε (see [25]). The values of a polynomial can be computed by Horner's scheme.

1. Computing the values of an exponential function in the interval [—1, 1]. The following polynomial approximation is used:

$$e^x \approx \sum_{k=0}^{7} a_k x^k \quad (|x| \leqslant 1), \quad \varepsilon = 2 \cdot 10^{-7}, \tag{1}$$

$a_0 = 0.9999998$, $a_1 = 1.0000000$, $a_2 = 0.5000063$, $a_3 = 0.1666674$,
$a_4 = 0.0416350$, $a_5 = 0.0083298$, $a_6 = 0.0014393$, $a_7 = 0.0002040$.

2. Computing the values of a logarithmic function. The following formula takes place

$$\ln(1+x) \approx \sum_{k=1}^{7} a_k x^k \ (0 \leqslant x \leqslant 1), \quad \varepsilon = 2.2 \cdot 10^{-7}, \tag{2}$$

$a_1 = 0.999981028$, $a_2 = -0.499470150$, $a_3 = 0.328233122$,
$a_4 = -0.225873284$, $a_5 = 0.134639267$, $a_6 = -0.055119959$,
$a_7 = 0.010757369$.

3. Computing the values of trigonometric functions. Use the following polynomial approximations:

$$\sin x \approx \sum_{k=0}^{4} a_{2k+1} x^{2k+1} \ (|x| \leqslant \pi/2), \quad \varepsilon = 6 \cdot 10^{-9}, \tag{3}$$

$a_1 = 1.000000002$, $a_3 = -0.166666589$, $a_5 = 0.008333075$,
$a_7 = -0.000198107$, $a_9 = 0.000002608$;

$$\cos x \approx \sum_{k=0}^{5} a_{2k} x^{2k} \ (|x| \leqslant 1), \quad \varepsilon = 2 \cdot 10^{-9}, \tag{4}$$

$a_0 = 1.000000000000$, $a_2 = -0.499999999942$,
$a_4 = 0.041666665950$, $a_6 = -0.001388885683$,
$a_8 = 0.000024795132$, $a_{10} = -0.000000269591$;

$$\tan x \approx \sum_{k=0}^{6} a_{2k+1} x^{2k+1} \ (|x| \leqslant \pi/4), \quad \varepsilon = 2 \cdot 10^{-8}, \tag{5}$$

$a_1 = 1.00000002$, $a_3 = 0.33333082$, $a_5 = 0.13339762$,
$a_7 = 0.05935836$, $a_9 = 0.02457096$, $a_{11} = 0.00294045$,
$a_{13} = 0.00947324$.

Example 1. Using a polynomial approximation, find the value of $\sqrt{e}$ to within 10^{-6}.

Solution. We carry out the computations by means of formula (1), using Horner's scheme (see Sec. 2.1) for $x=0.5$ (see Table 2).

Table 2

Horner's Scheme for Polynomial (1)

0.0002040	0.0014393	0.0083298	0.0416350
+	0.0001020	0.0007706	0.0045502
0.0002040	0.0015413	0.0091004	0.0461852

0.1666674	0.5000063	1.0000000	0.9999998 | 0.5
0.0230926	0.0948800	0.2974431	0.6487216
0.1897600	0.5948863	1.2974431	1.6487214 = P (0.5)

Rounding off to six digits, we get $e^{1/2} \approx 1.648721$ (see Example 1 in Sec. 2.2).

Example 2. Using the polynomial approximation, find the value of sin 0.5 to within 10^{-8}.

Solution. We can carry out the computations by formula (3), using Horner's scheme for $x=0.5$. But since the polynomial of formula (3) contains only odd powers of x, it is more convenient to rewrite it in the form

$$\sum_{k=0}^{4} a_{2k+1}x^{2k+1} = (a_1 + a_3x^2 + a_5x^4 + a_7x^6 + a_9x^8)\,x$$

and apply Horner's scheme to the polynomial

$$P(\xi) = a_1 + a_3\xi + a_5\xi^2 + a_7\xi^3 + a_9\xi^4, \quad \xi = x^2 = 0.25. \qquad (6)$$

The computation of the value $P(0.25)$ is given in Table 3. Multiplying the obtained value $P(0.25) = 0.958851087$ by $x=0.5$ and rounding off the result, we get the required value $\sin 0.5 \approx \approx 0.47942554$.

Table 3

Horner's Scheme for Polynomial (6)

+0.000002608	—0.000198107	0.008333075	—0.166666589	1.000000002 | 0.25
	0.000000652	—0.000049364	0.002070928	—0.041148915
0.000002608	—0.000197445	0.008283711	—0.164595661	0.958851087 = P(0.25)

Problems

1. Form the tables of the values of the following functions accurate to ε for the given values of x.

(a) e^x, $x=0.725+0.001k$ $(k=0, 1, 2, \ldots, 15)$, $\varepsilon=10^{-4}$, $x=0.213+0.002k$ $(k=0, 1, 2, \ldots, 15)$, $\varepsilon=10^{-5}$,

(b) e^{-x}, $x=0.213+0.003k$ $(k=0, 1, 2, \ldots, 15)$, $\varepsilon=10^{-5}$, $x=1.27+0.02k$ $(k=0, 1, 2, \ldots, 15)$, $\varepsilon=10^{-5}$,

(c) $e^{1/x}$, $x=2+0.5k$ $(k=0, 1, 2, \ldots, 15)$, $\varepsilon=10^{-5}$,

(d) $\frac{1}{2\sqrt{\pi}}e^{-x^2/2}$, $x=0.4+0.02k$ $(k=0, 1, 2, \ldots, 15)$, $\varepsilon=10^{-5}$, $x=1.2+0.1k$ $(k=0, 1, 2, \ldots, 15)$, $\varepsilon=10^{-5}$.

2. Form the tables for the following functions accurate to 10^{-5} for the indicated values of x.

(a) $\sin x$, $x=0.055+0.003k$ $(k=0, 1, \ldots, 15)$, $x=0.80+0.05k$ $(k=0, 1, \ldots, 15)$,

(b) $\cos x$ for the same values of x, (c) $\tan x$ for the same values of x.

2.4 USING CONTINUED FRACTIONS FOR COMPUTING THE VALUES OF TRANSCENDENTAL FUNCTIONS

1. Introductory notes. Let $a_0, a_1, a_2, \ldots, a_n, \ldots$, and $b_1, b_2, \ldots, b_n, \ldots$ be two sequences. An expression of the form

$$a_0+\cfrac{b_1}{a_1+\cfrac{b_2}{a_2+\cfrac{b_3}{a_3+\ddots}}}=\left[a_0;\ \frac{b_1}{a_1},\ \frac{b_2}{a_2},\ \frac{b_3}{a_3},\ \ldots\right] \tag{1}$$

is called *a continued fraction* corresponding to the given sequences $\{a_k\}_{k=0}^{\infty}$, $\{b_k\}_{k=1}^{\infty}$. In the general case, the elements of a continued fraction a_0, a_k, b_k $(k=1, 2, \ldots)$ are real or complex numbers, or functions of one or more variables. The expressions

$$a_0+\frac{b_1}{a_1},\quad a_0+\cfrac{b_1}{a_1+\cfrac{b_2}{a_2}},\quad a_0+\cfrac{b_1}{a_1+\cfrac{b_2}{a_2+\cfrac{b_3}{a_3}}},\quad \ldots,\quad a_0+\cfrac{b_1}{a_1+\cfrac{b_2}{a_2+\ddots}}$$

$$\ddots$$

$$a_{n-1}+\frac{b_n}{a_n}$$

are called, respectively, the first, second, third, ..., nth *convergents* of the given continued fraction, and are usually denoted $\frac{P_1}{Q_1}, \frac{P_2}{Q_2}, \ldots, \frac{P_n}{Q_n}$. When using a digital computer the convergents

are conveniently found with the aid of Horner's scheme for division:

$$c_1 = \frac{b_n}{a_n}, \qquad d_1 = a_{n-1} + c_1,$$
$$c_2 = \frac{b_{n-1}}{d_1}, \qquad d_2 = a_{n-2} + c_2,$$
$$\ldots\ldots$$
$$c_k = \frac{b_{n-k+1}}{d_{k-1}}, \qquad d_k = a_{n-k} + c_k,$$
$$\ldots\ldots\ldots\ldots\ldots\ldots$$
$$c_n = \frac{b_1}{d_{n-1}}, \qquad d_n = a_0 + c_n = \frac{P_n}{Q_n}.$$

The indicated sequence of operations can readily be programmed.

For the numerators and denominators of the convergents there exist the following recurrent formulas:

$$P_n = a_n P_{n-1} + b_n P_{n-2}, \quad Q_n = a_n Q_{n-1} + b_n Q_{n-2} \ (n = 1, 2, \ldots),$$

where $P_{-1} = 1$, $Q_{-1} = 0$, $P_0 = a_0$, $Q_0 = 1$.

The continued fraction (1) is called *convergent* if there exists a finite limit A of the convergent P_n/Q_n as $n \to \infty$:

$$A = \lim_{n \to \infty} \frac{P_n}{Q_n},$$

the number A being taken for the value of this fraction. If the limit does not exist, then the continued fraction (1) is called *divergent*. For a convergent continued fraction the convergent $\frac{P_n}{Q_n}$ is its approximate value.

2. Expanding e^x into a continued fraction. Use the following expansion

$$e^x = \left[0, \frac{1}{1}, \frac{-2x}{2+x}, \frac{x^2}{6}, \frac{x^2}{10}, \ldots, \frac{x^2}{4n+2}, \ldots\right] \tag{2}$$

which converges for any value of x. From this we obtain the convergents

$$\frac{P_1}{Q_1} = \frac{1}{1}, \quad \frac{P_2}{Q_2} = \frac{2+x}{2-x}, \quad \frac{P_3}{Q_3} = \frac{12+6x+x^2}{12-6x+x^2},$$
$$\frac{P_4}{Q_4} = \frac{120+60x+12x^2+x^3}{120-60x+12x^2-x^3},$$
$$\frac{P_5}{Q_5} = \frac{1680+840x+180x^2+20x^3+x^4}{1680-840x+180x^2-20x^3+x^4}.$$

Example 1. Using the expansion of e^x into a continued fraction, compute $\frac{1}{e}$ accurate to 10^{-5}.

Solution. Let us compute the fourth and fifth convergents for $x=-1$:

$$\frac{P_4}{Q_4}=\frac{120-60+12-1}{120+60+12+1}=\frac{71}{193}\approx 0.367876,$$

$$\frac{P_5}{Q_5}=\frac{1680-840+180-20+1}{1680+840+180+20+1}=\frac{1001}{2721}\approx 0.367879.$$

Comparing the obtained approximations, we note that their five decimal digits coincide; therefore we may put $e^{-1}\approx 0.36788$.

3. Expanding tan x into a continued fraction. There exists the following expansion:

$$\tan x=\left[0,\ \frac{x}{1},\ -\frac{x^2}{3},\ -\frac{x^2}{5},\ \ldots,\ -\frac{x^2}{2n+1},\ \ldots\right]. \qquad (3)$$

This expansion holds true for all the points of continuity of $\tan x$. The first convergents will be:

$$\frac{P_1}{Q_1}=\frac{x}{1},\qquad \frac{P_4}{Q_4}=\frac{105x-10x^3}{105-45x^2+x^4},$$

$$\frac{P_2}{Q_2}=\frac{3x}{3-x^2},\qquad \frac{P_5}{Q_5}=\frac{945x-105x^3+x^5}{945-420x^2+15x^4},$$

$$\frac{P_3}{Q_3}=\frac{15x-x^3}{15-6x^2},$$

Problems

1. Using expansions into a continued fraction, compile the tables of the values of the following functions accurate to ε:

(a) e^x, $x=0.155+0.005k$ $(k=0,\ 1,\ 2,\ \ldots,\ 15)$, $\varepsilon=10^{-5}$, $x=-0.30+0.03k$ $(k=0,\ 1,\ 2,\ \ldots,\ 15)$, $\varepsilon=10^{-4}$,

(b) $\tan x$, $x=0.47+0.01k$ $(k=0,\ 1,\ 2,\ \ldots,\ 15)$, $\varepsilon=10^{-5}$.

2.5 USING THE METHOD OF ITERATION FOR APPROXIMATING THE VALUES OF A FUNCTION

Any function $y=f(x)$ can be represented in different ways in implicit form, i.e. by an equation

$$F(x,\ y)=0. \qquad (1)$$

It often happens that the solution of equation (1) with respect to y using the method of iteration is reduced to uniform operations easily performed on a computer. Then, obviously, it is advisable to use the method of iteration.

One of possible iteration processes for computing $y(x)$ can be formed as follows.

Let y_n be an approximate value of y. Using Lagrange's formula, we get

$$F(x, y_n) = F(x, y_n) - F(x, y) = (y_n - y) F'_y(x, \bar{y}_n),$$

where $\bar{y}_n$ is some intermediate value between y_n and y. Hence

$$y = y_n - \frac{F(x, y_n)}{F'_y(x, \bar{y}_n)},$$

the value $\bar{y}_n$ being not known to us.

Putting approximately $\bar{y}_n \approx y_n$, we get the following formula for computing $y \approx y_{n+1}$:

$$y_{n+1} = y_n - \frac{F(x, y_n)}{F'_y(x, y_n)} \quad (n = 0, 1, 2, \ldots). \tag{2}$$

The proposed iteration process is the Newton method (see [12]) applied to the function $z = F(x, y)$ with x fixed, therefore its convergence will be, generally speaking, ensured if $F'_y(x, y)$ and $F''_{yy}(x, y)$ exist and retain constant signs in the interval that contains the root of $y(x)$.

The initial approximation $y_0(x)$ is chosen so that it is readily computed and is as close to the true value of $y(x)$ as possible. The process of iteration is continued until, within the limits of the given accuracy, two successive values y_{n-1} and y_n coincide. Then we approximately put $y(x) \approx y_{n+1}$.

1. Computing reciprocals. Let $y = \frac{1}{x}$ $(x > 0)$. Set

$$F(x, y) \equiv x - \frac{1}{y} = 0.$$

Using formula (2), we get

$$y_{n+1} = y_n (2 - x y_n). \tag{3}$$

We have thus obtained an iterative process without division. The initial value y_0 is usually chosen in the following manner. Suppose the argument x is written in binary notation:

$$x = 2^m x_1,$$

where m is an integer and $\frac{1}{2} \leqslant x_1 < 1$. Then set $y_0 = 2^{-m}$; with such a choice of the initial value y_0 the convergence of the iterative process is rather rapid (see [12]).

Note. Since the quotient $\frac{a}{b}$ is the product of a by $\frac{1}{b}$, we can perform the operation of division on computers not intended for this operation in two stages:

(1) computation of $y = \frac{1}{b}$ (the reciprocal of the divisor),

(2) multiplication of y by the dividend a.

Example 1. With the aid of formula (3) find the value of the function $y = \frac{1}{x}$ for $x = 5$ accurate to 10^{-4}.

Solution. Write the argument x in the form $x = 2^3 \cdot \frac{5}{8}$. Put $y_0 = 2^{-3} = 1/8$. Then by formula (3), we have

$$y_1 = \frac{1}{8}\left(2 - \frac{5}{8}\right) = \frac{11}{64} = 0.1718, \qquad y_2 = \frac{11}{64}\left(2 - \frac{55}{64}\right) = \frac{803}{4096} = 0.1960,$$

$$y_3 = 0.1960\,(2 - 0.9800) = 0.1960 \cdot 1.0200 = 0.19992.$$

We see that here already the third approximation yields $y(x) \approx 0.1999$ accurate to 10^{-4}.

2. Computing square roots. Let $y = \sqrt{x}$ $(x > 0)$. Bring this equation to the form

$$F(x, y) = y^2 - x = 0.$$

Using formula (2) we have

$$y_{n+1} = \frac{1}{2}\left(y_n + \frac{x}{y_n}\right) \quad (n = 0, 1, 2, \ldots), \tag{4}$$

which is *Hero's formula* (*Hero's process*).

Let the argument x be written in binary notation:

$$x = 2^m x_1,$$

where m is an integer and $1/2 \leqslant x_1 < 1$. Then they usually put

$$y_0 = 2^{E(m/2)},$$

where $E(m/2)$ is an integral part of the number $m/2$.

The iteration process using Hero's formula is easily realized on a computer having the division as one of elementary operations; in this case the iteration process converges at any choice of $y_0 > 0$ (in this example it is easy to check how the above mentioned conditions of convergence are met, since $F'_y = 2y > 0$ and $F''_{yy} = 2 > 0$).

If $0.01 \leqslant x \leqslant 1$, then we can take $y_0 = ax + b$ for the initial approximation; the appropriate coefficients a and b are given in Table 4.

Table 4

Coefficients for Initial Approximation in Hero's Formula (4)

Interval	a	b	Interval	a	b
(0.01; 0.02)	4.1	0.060	(0.18; 0.30)	1.0	0.247
(0.02; 0.03)	3.2	0.078	(0.30; 0.60)	0.8	0.304
(0.03; 0.08)	2.2	0.110	(0.60; 1.00)	0.6	0.409
(0.08; 0.18)	1.4	0.174			

At such a choice of the initial approximation y_0 the second iteration y_2 already gives the value of $\sqrt{x}$ with eight decimal digits; when computing y_0 the value of x may be taken only with three decimal digits (see [25]).

Example 2. Find $\sqrt{7}$ to within 10^{-5}.

Solution. Here $x = 7 \cdot 2^{-4} \cdot \frac{7}{8}$. Hence, the initial approximation has the form

$$y_0 = 2^{E(3/2)} = 2.$$

By formula (4) we successively find

$$y_1 = \frac{1}{2}\left(2 + \frac{7}{2}\right) = 2.75000,$$

$$y_2 = \frac{1}{2}\left(\frac{11}{4} + \frac{28}{11}\right) = 2.64772,$$

$$y_3 = \frac{1}{2}\left(\frac{233}{88} + \frac{616}{233}\right) = 2.64575,$$

$$y_4 = \frac{1}{2}\left(2.64575 - \frac{7}{2.64575}\right) = 2.64575.$$

Since in the values y_3 and y_4 five decimal digits coincide, we take $\sqrt{7} \approx 2.64575$.

Note. If computation is performed on a computer whose system of commands does not contain the operation of division, then we can use another iteration formula, and namely:

$$y_{n+1} = y_n\left(\frac{3}{2} - \frac{y_n^2}{2x}\right) \quad (n = 0, 1, 2, \ldots). \tag{5}$$

According to this formula extracting a square root is reduced to single computation of the reciprocal $\frac{1}{2x}$ and then to an iteration process, each stage of which incorporates only the operations of multiplication and subtraction. Formula (5) corresponds to the transformation of the initial equation to the form $F(x, y) = \frac{1}{y^2} - \frac{1}{x} = 0$.

3. Computing the reciprocal of a square root. Let us have

$$y = \frac{1}{\sqrt{x}}, \quad (x > 0).$$

The iterative formula for computing the reciprocal of a square root has the form

$$y_{n+1} = \frac{3}{2} y_n - \frac{1}{2} x y_n^3 \quad (n = 0, 1, 2, \ldots). \tag{6}$$

This formula is obtained by transforming the initial equation $y=\frac{1}{\sqrt{x}}$ to the form $F(x, y)\equiv\frac{1}{y^2}-x=0$.

As the initial approximation we usually take

$$y_0=2^{-E\left(\frac{m}{2}\right)},$$

where $x=2^m x_1$, $\frac{1}{2}\leqslant x_1<1$.

Here we also have an iteration process without division.

4. Computing cube roots. Let us have $y=\sqrt[3]{x}$. Applying formula (2) to the equation $F(x, y)\equiv y^3-x=0$, we get an iteration formula for computing the cube root in the form

$$y_{n+1}=\frac{1}{3}\left(\frac{2y_n^3+x}{y_n^2}\right); \tag{7}$$

the initial approximation

$$y_0=2^{E(m/3)},$$

where $x=2^m x_1$, m is an integer and $1/2\leqslant x_1<1$.

Example 3. Compute $\sqrt[3]{5}$ to within 10^{-3}.

Solution. Here $x=5=2^3\cdot\frac{5}{8}$. The initial approximation

$$y_0=2^{E(3/3)}=2,$$

$$y_1=\frac{1}{3}\left(4+\frac{5}{4}\right)=\frac{21}{12}=1.7500.$$

The further computations are tabulated below.

Table 5

Computing $\sqrt[3]{5}$

n	y_n	y_n^2	$3y_n^2$	y_n^3	$2y_n^3+5$
0	2	4	12	8	21
1	1.7500	3.0625	9.1875	5.3594	15.7188
2	1.7100	2.9241	8.7723	5.0002	15.0004
3	1.7100				

Thus $\sqrt[3]{5}\approx 1.710$.

5. Computing roots of degree p. Let

$$y=\sqrt[p]{x},$$

where $x>0$ and $p>0$ is an integer.

Applying formula (2) to the equation $F(x, y) \equiv 1 - \frac{x}{y^p} = 0$, we get

$$y_{n+1} = y_n \left[\left(1 + \frac{1}{p} \right) - \frac{y_n^p}{px} \right]. \tag{8}$$

The iteration process will be convergent if the initial approximation $y_0 > 0$ is chosen so small that $y_0^p < (p+1)x$.

6. Newton's formula for computing roots of degree p. Let $y = \sqrt[p]{x}$; then the following formula takes place

$$y_{n+1} = \frac{1}{p} \left[(p-1) y_n + \frac{x}{y_n^{p-1}} \right], \tag{9}$$

obtained from formula (2) for $F(x, y) \equiv y^p - x$.

The initial approximation y_0 can be chosen accurate to one or two significant digits.

For $p = 2$ Newton's formula yields Hero's formula.

Example 4. Compute $y = \sqrt[7]{277\,234}$ to within 10^{-6}.

Solution. Let us take $y_0 = 6$; using formula (8) we successively compute:

$$y_1 = 6 \left[\left(1 + \frac{1}{7} \right) - \frac{6^7}{7 \cdot 277\,234} \right] = 5.99164605,$$

$$y_2 = 5.99169225, \quad y_3 = 5.99169225.$$

Answer: $\sqrt[7]{277\,234} \approx 5.991692$.

Problems

1. Using the method of iteration, form the tables of the values of the following functions accurate to 10^{-6}:

(a) $\frac{1}{x}$, $x = 3 + 2k$ $(k = 0, 1, 2, \ldots, 15)$, (b) $\frac{1}{x^2}$ for the same values of x, (c) $\frac{1}{x^3}$ for the same values of x, (d) $\frac{x}{1+x}$, $x = 0.007 + 0.003k$ $(k = 0, 1, 2, \ldots, 15)$.

2. Using the method of iteration, form the tables of the values of the following functions accurate to 10^{-5}:

(a) $\sqrt{x}$, $x = 2 + k$ $(k = 0, 1, 2, \ldots, 15)$,

(b) $x\sqrt{x}$ for the same values of x,

(c) $\sqrt{1+x^2}$, $x = 0.3 + 0.002k$ $(k = 0, 1, 2, \ldots, 15)$,

(d) $\sqrt{x^2+1}/x$ for the same values of x.

3. Using the method of iteration, form the tables of the values of the following functions accurate to 10^{-3}:

(a) $\frac{1}{\sqrt{x}}$, $x = 3 + 2k$ $(k = 0, 1, 2, \ldots, 15)$,

(b) $1/\sqrt{2+x^2}$, $x=0.3+0.002k$ $(k=0, 1, 2, \ldots, 15)$,

(c) $(2x+1)/\sqrt{x}$, $x=3.1+0.005k$ $(k=0, 1, 2, \ldots, 15)$,

(d) $1/\sqrt{x(x+1)}$, $x=2.3+0.002k$ $(k=0, 1, 2, \ldots, 15)$.

4. Using the method of iteration, form the tables of the values of the following functions accurate to 10^{-6}:

(a) $\sqrt[3]{x}$, $x=3+k$ $(k=0, 1, 2, \ldots, 15)$,

(b) $\frac{1}{\sqrt[3]{x}}$ for the same values of x.

5. Using the method of iteration, form the tables of the values of the following functions accurate to 10^{-6}:

(a) $\sqrt[4]{x}$, $x=0.05+0.02k$ $(k=0, 1, 2, \ldots, 15)$,

(b) $\sqrt[5]{x}$ for the same values of x,

(c) $\sqrt[6]{x}$ for the same values of x,

(d) $\sqrt[7]{x}$ for the same values of x.

3

NUMERICAL SOLUTION OF SYSTEMS OF LINEAR ALGEBRAIC EQUATIONS

3.1 BASIC CONCEPTS

Suppose we have a system of n linear algebraic equations in n unknowns

$$\left.\begin{aligned} a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n &= b_1, \\ a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n &= b_2, \\ \cdots\cdots\cdots\cdots\cdots\cdots & \\ a_{n1}x_1 + a_{n2}x_2 + \ldots + a_{nn}x_n &= b_n, \end{aligned}\right\} \tag{1}$$

or in the matrix form

$$A\boldsymbol{x} = \boldsymbol{b}, \tag{2}$$

where

$$A = (a_{ij}) = \begin{pmatrix} a_{11} & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{22} & \ldots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & \ldots & a_{nn} \end{pmatrix}$$

is the matrix of the coefficients, and

$$\boldsymbol{b} = \begin{pmatrix} b_1 \\ b_2 \\ \ldots \\ b_n \end{pmatrix}, \quad \boldsymbol{x} = \begin{pmatrix} x_1 \\ x_2 \\ \ldots \\ x_n \end{pmatrix}$$

are the columns of the constant terms and the unknowns, respectively.

If the matrix A is nonsingular, that is

$$\det A = \begin{vmatrix} a_{11} & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{22} & \ldots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & \ldots & a_{nn} \end{vmatrix} \neq 0,$$

then system (1) has the unique solution. In this case the solution of system (1) from the theoretical point of view is not a difficult thing. The values of the unknowns x_i ($i = 1, 2, \ldots, n$) can be obtained using Cramer's formulas

$$x_i = \frac{\det A_i}{\det A},$$

where the matrix A_i is obtained from the matrix A by replacing its ith column by the column of the constant terms.

But such a method of solving a linear system in n unknowns leads to computing $n+1$ determinants of the nth order, which is quite a laborious operation especially when the number n is rather large.

Methods of solving systems of linear equations applied at present are divided mainly into two groups: exact and iterative.

Exact methods are defined as methods, which, in assumption that the computations are carried out exactly (without roundings), yield exact values of the unknowns x_i. Because of unavoidable rounding in practical applications, the results of even the exact methods are approximate, and error estimates of the roots are involved. Examples of exact methods are the Gaussian method, the method of square roots, etc.

Iterative methods include such methods which, even in assumption that the computations are performed without rounding, permit obtaining the roots of a system $(x_1, x_2, \ldots, x_n)$ only to a preassigned accuracy. The exact solution of a system in these cases can be obtained theoretically as a result of an infinite process. This group includes the method of simple iteration, the Seidel method, the method of relaxation, and others. Each of these methods is not always convergent as applied to a concrete class of systems of linear equations.

3.2 THE GAUSSIAN METHOD

The most common technique for the solution of systems of linear algebraic equations is via an algorithm for the successive elimination of the unknowns. This method is called the Gaussian method. There are various computational schemes realizing this method. Let us consider one of them—the *scheme of unique division*.

For the sake of simplicity, we confine ourselves to a system of four equations in four unknowns:

$$\left.\begin{aligned} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + a_{14}x_4 &= a_{15}, \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + a_{24}x_4 &= a_{25}, \\ a_{31}x_1 + a_{32}x_2 + a_{33}x_3 + a_{34}x_4 &= a_{35}, \\ a_{41}x_1 + a_{42}x_2 + a_{43}x_3 + a_{44}x_4 &= a_{45}. \end{aligned}\right\} \qquad (1)$$

Let $a_{11} \neq 0$ (the leading element). Dividing the first equation of system (1) by a_{11}, we get

$$x_1 + b_{12}x_2 + b_{13}x_3 + b_{14}x_4 = b_{15}, \tag{2}$$

where

$$b_{1j} = \frac{a_{1j}}{a_{11}} \quad (j = 2, 3, 4, 5).$$

Using equation (2), it is easy to eliminate the unknown x_1 from the second, third, and fourth equations of the system (1). To do this, multiply equation (2) by a_{21}, a_{31} and a_{41}, and subtract the results thus obtained from the second, third, and fourth equations of the system, respectively. We finally get a system consisting of three equations:

$$\left.\begin{aligned} a_{22}^{(1)}x_2 + a_{23}^{(1)}x_3 + a_{24}^{(1)}x_4 &= a_{25}^{(1)}, \\ a_{32}^{(1)}x_2 + a_{33}^{(1)}x_3 + a_{34}^{(1)}x_4 &= a_{35}^{(1)}, \\ a_{42}^{(1)}x_2 + a_{43}^{(1)}x_3 + a_{44}^{(1)}x_4 &= a_{45}^{(1)}, \end{aligned}\right\} \tag{3}$$

where the coefficients $a_{ij}^{(1)}$ are computed from the formula

$$a_{ij}^{(1)} = a_{ij} - a_{i1}b_{1j} \quad (i = 2, 3, 4; \quad j = 2, 3, 4, 5). \tag{4}$$

Now, dividing the first equation of system (3) by the leading element $a_{22}^{(1)}$, we get the equation

$$x_2 + b_{23}^{(1)}x_3 + b_{24}^{(1)}x_4 = b_{25}^{(1)}, \tag{5}$$

where

$$b_{2j}^{(1)} = \frac{a_{2j}^{(1)}}{a_{22}^{(1)}} \quad (j = 3, 4, 5).$$

Eliminating x_2 in the same way that we eliminated x_1, we arrive at the following system of equations

$$\left.\begin{aligned} a_{33}^{(2)}x_3 + a_{34}^{(2)}x_4 &= a_{35}^{(2)}, \\ a_{43}^{(2)}x_3 + a_{44}^{(2)}x_4 &= a_{45}^{(2)}, \end{aligned}\right\} \tag{6}$$

where

$$a_{ij}^{(2)} = a_{ij}^{(1)} - a_{i2}^{(1)}b_{2j}^{(1)} \quad (i = 3, 4; \quad j = 3, 4, 5). \tag{7}$$

Dividing the first equation of (6) by the leading element $a_{33}^{(2)}$, we get

$$x_3 + b_{34}^{(2)}x_4 = b_{35}^{(2)}, \tag{8}$$

where $b_{3j}^{(2)} = \dfrac{a_{3j}^{(2)}}{a_{33}^{(2)}}$ $(j = 4, 5)$.

Then with the aid of this equation we eliminate x_3 from the second equation of (6). We get the equation

$$a_{44}^{(3)}x_4 = a_{45}^{(3)},$$

where

$$a_{4j}^{(3)} = a_{4j}^{(2)} - a_{43}^{(2)}b_{3j}^{(2)} \quad (j = 4, 5). \tag{9}$$

Thus, we have reduced system (1) to an equivalent system with a *triangular matrix*:

$$\left.\begin{aligned} x_1+b_{12}x_2+b_{13}x_3+b_{14}x_4&=b_{15},\\ x_2+b_{23}^{(1)}x_3+b_{24}^{(1)}x_4&=b_{25}^{(1)},\\ x_3+b_{34}^{(2)}x_4&=b_{35}^{(2)},\\ a_{44}^{(3)}x_4&=a_{45}^{(3)}, \end{aligned}\right\} \tag{10}$$

whence we successively find

$$\left.\begin{aligned} &x_4=a_{45}^{(3)}/a_{44}^{(3)}, \quad x_3=b_{35}^{(2)}-b_{34}^{(2)}x_4,\\ &x_2=b_{25}^{(1)}-b_{24}^{(1)}x_4-b_{23}^{(1)}x_3,\\ &x_1=b_{15}-b_{14}x_4-b_{13}x_3-b_{12}x_2. \end{aligned}\right\} \tag{11}$$

Thus, the process of solution comprises two procedures:

(1) *direct procedure*, i. e. reducing the system (1) to the triangular form (10);

(2) *reverse procedure*, i. e. finding the values of the unknowns by the formulas (11).

As is obvious, the above considered method is applied only if all the leading elements are nonzero. If one of them vanishes, it is sufficient to change places of the equations in the corresponding system to make the leading element nonzero (of course, assuming that the matrix A is nonsingular).

The number N of arithmetic operations necessary to realize the Gaussian method is determined by the following formula (see [2], [12]):

$$N=\frac{2n\,(n+1)\,(n+2)}{3}+n\,(n-1),$$

where n is the number of unknowns.

Thus, the time required for the performance of arithmetic operations in solving a system of linear algebraic equations by the Gaussian method is approximately proportional to the cube of the number of the unknowns.

Example 1. Using the Gaussian method, solve the following system of equations

$$\left.\begin{aligned} 2.0x_1+1.0x_2-0.1x_3+1.0x_4&=2.7,\\ 0.4x_1+0.5x_2+4.0x_3-8.5x_4&=21.9,\\ 0.3x_1-1.0x_2+1.0x_3+5.2x_4&=-3.9,\\ 1.0x_1+0.2x_2+2.5x_3-1.0x_4&=9.9. \end{aligned}\right\} \tag{12}$$

Solution. Direct procedure. Dividing the first equation of system (12) by $a_{11}=2$, we get

$$x_1+0.5x_2-0.05x_3+0.5x_4=1.35.$$

Hence, $b_{12}=0.5$; $b_{13}=-0.05$; $b_{14}=0.5$; $b_{15}=1.35$. Using formula (4), compute the coefficients $a_{ij}^{(1)}$ and form system (3). For

$i=2$ we have

$$a_{22}^{(1)}=a_{22}-a_{21}b_{12}=0.5-0.4\cdot 0.5=0.3,$$
$$a_{23}^{(1)}=a_{23}-a_{21}b_{13}=4+0.4\cdot 0.05=4.02,$$
$$a_{24}^{(1)}=a_{24}-a_{21}b_{14}=-8.5-0.4\cdot 0.5=-8.7,$$
$$a_{25}^{(1)}=a_{25}-a_{21}b_{15}=21.9-0.4\cdot 1.35=21.36.$$

For $i=3$, 4 computations are performed in a similar way. Thus, we get a system in three unknowns:

$$\begin{aligned}0.3x_2+4.02x_3-8.7x_4&=21.36,\\ -1.15x_2+1.015x_3+5.05x_4&=-4.305,\\ -0.3x_2+2.55x_3-1.5x_4&=8.55.\end{aligned}$$

Dividing the first equation of the obtained system by $a_{22}^{(1)}=0.3$, we write:

$$x_2+13.40x_3-29.00x_4=71.20,$$

where $b_{23}^{(1)}=13.40$, $b_{24}^{(1)}=-29.00$, $b_{25}^{(1)}=71.20$.

Using formulas (7), compute the coefficients $a_{ij}^{(2)}$ and form system (6). For $i=3$ we have

$$a_{33}^{(2)}=a_{33}^{(1)}-a_{32}^{(1)}b_{23}^{(1)}=1.015+1.15\cdot 13.40=16.425,$$
$$a_{34}^{(2)}=a_{34}^{(1)}-a_{32}^{(1)}b_{24}^{(1)}=5.05-1.15\cdot 29.00=-28.300,$$
$$a_{35}^{(2)}=a_{35}^{(1)}-a_{32}^{(1)}b_{25}^{(1)}=-4.305+1.15\cdot 71.20=77.575.$$

For $i=4$ computations are performed in a similar way. Write a system in two unknowns:

$$\begin{aligned}16.425x_3-28.300x_4&=77.575,\\ 6.570x_3-10.200x_4&=29.910.\end{aligned}$$

Dividing the first equation of this system by $a_{33}^{(2)}=16.425$, we get

$$x_3-1.72298x_4=4.72298,$$

where $b_{34}^{(2)}=-1.72298$; $b_{35}^{(2)}=4.72298$.

Using formulas (9), find the coefficients $a_{4j}^{(3)}$:

$$a_{44}^{(3)}=a_{44}^{(2)}-a_{43}^{(2)}b_{34}^{(2)}=-10.200+6.570\cdot 1.72298=1.11998,$$
$$a_{45}^{(3)}=a_{45}^{(2)}-a_{43}^{(2)}b_{35}^{(2)}=29.910-6.570\cdot 4.72298=-1.11998,$$

and write one equation in one unknown:

$$1.11998x_4=-1.11998.$$

Thus, the equivalent system has the form

$$\left.\begin{aligned}x_1+0.5x_2-0.05x_3+0.5x_4&=1.35,\\ x_2+13.40x_3-29.00x_4&=71.20,\\ x_3-1.72298x_4&=4.72298,\\ 1.11998x_4&=-1.11998.\end{aligned}\right\}\qquad(13)$$

Here the direct procedure is terminated.

Reverse procedure. From system (13) we successively find

$$x_4 = -1.00000,$$
$$x_3 = 4.72298 - 1.72298 = 3.00000,$$
$$x_2 = 71.20 - 13.40 \cdot 3 + 29.0 = 2.00000,$$
$$x_1 = 1.35 - 0.5 \cdot 2 + 0.05 \cdot 3 + 0.5 = 1.00000.$$

Since we are not guaranteed from errors, all computations should be checked. One of the simplest check schemes is based on the fact that an increase of the values of all unknowns by unity is equivalent to replacement of the given system (1) by the check one, in which the constant terms are equal to the sums of all the coefficients of the corresponding row, for instance:

$$a_{16} = a_{11} + a_{12} + a_{13} + a_{14} + a_{15}.$$

Solving the check system simultaneously with the given one, we acquire the possibility to check each operation in computing the given system as it will be shown below on concrete schemes.

3.3 THE GAUSSIAN COMPACT SCHEME. THE MODIFICATION OF CROUT-DOOLITTLE

1. The Gaussian compact scheme. If computations are performed on a key computer by means of the scheme of unique division, then much time is spent on recording intermediate results. The Gaussian compact scheme (see [2]) provides an economical method of recording. Let us consider the sequence of forming this scheme for system (1) of Sec. 3.2. All the results of computations we shall enter into one table (Table 6).

The table is filled in in the following order.

Direct procedure:

(1) Write down the coefficients of the given system in four rows and five columns of Section I of Table 6.

(2) Sum all the coefficients of the row and write down the sum in the $\sum$ column (the check column), for instance, $a_{16} = \sum_{j=1}^{5} a_{1j}$.

(3) Divide all the numbers of the first row by a_{11} and enter the results $b_{1j} = \frac{a_{1j}}{a_{11}}$ into the fifth row of Section I.

(4) Compute $\sum_{j=1}^{5} b_{1j}$ and carry out a check. If the computations are performed with a constant number of decimal digits, then the numbers b_{16} and $\sum_{j=1}^{5} b_{1j}$ should not differ by more than one unit of the last retained digit, otherwise, check the operations of (3).

Table 6
The Gaussian Compact Scheme

	i	a_{i1}	a_{i2}	a_{i3}	a_{i4}	a_{i5}	$=a_{i6}$
I	1	a_{11}	a_{12}	a_{13}	a_{14}	a_{15}	$\sum a_{1j}=a_{16}$
	2	a_{21}	a_{22}	a_{23}	a_{24}	a_{25}	$\sum a_{2j}=a_{26}$
	3	a_{31}	a_{32}	a_{33}	a_{34}	a_{35}	$\sum a_{3j}=a_{36}$
	4	a_{41}	a_{42}	a_{43}	a_{44}	a_{45}	$\sum a_{4j}=a_{46}$
		1	b_{12}	b_{13}	b_{14}	b_{15}	$a_{16}/a_{11}=b_{16}$
II	2		$a_{22}^{(1)}$	$a_{23}^{(1)}$	$a_{24}^{(1)}$	$a_{25}^{(1)}$	$a_{26}^{(1)}$
	3		$a_{32}^{(1)}$	$a_{33}^{(1)}$	$a_{34}^{(1)}$	$a_{35}^{(1)}$	$a_{36}^{(1)}$
	4		$a_{42}^{(1)}$	$a_{43}^{(1)}$	$a_{44}^{(1)}$	$a_{45}^{(1)}$	$a_{46}^{(1)}$
			1	$b_{23}^{(1)}$	$b_{24}^{(1)}$	$b_{25}^{(1)}$	$a_{26}^{(1)}/a_{22}^{(1)}=b_{26}^{(1)}$
III	3			$a_{33}^{(2)}$	$a_{34}^{(2)}$	$a_{35}^{(2)}$	$a_{36}^{(2)}$
	4			$a_{43}^{(2)}$	$a_{44}^{(2)}$	$a_{45}^{(2)}$	$a_{46}^{(2)}$
				1	$b_{34}^{(2)}$	$b_{35}^{(2)}$	$a_{36}^{(2)}/a_{33}^{(2)}=b_{36}^{(2)}$
IV	4				$a_{44}^{(3)}$	$a_{45}^{(3)}$	$a_{46}^{(3)}$
V					1	x_4	$\bar{x}_4$
				1		x_3	$\bar{x}_3$
			1			x_2	$\bar{x}_2$
		1				x_1	$\bar{x}_1$

(5) Using formula (4) in Sec. 3.2, compute the coefficients

$$a_{ij}^{(1)} \quad (i=2,\ 3,\ 4; \quad j=2,\ 3,\ 4,\ 5,\ 6).$$

As a rule, key computers permit carrying out computations by formula (4) of Sec. 3.2 without recording the product $a_{i1}b_{1j}$. The results are written down in the first three rows of Section II.

(6) Make a check. The sum of the elements of each row $\sum_{j=2}^{5} a_{ij}^{(1)}$ $(i=2,\ 3,\ 4)$ must not differ from $a_{i6}^{(1)}$ by more than one

unit of the last retained digit (if all the computations are carried out with constant number of decimal digits).

(7) Divide all elements of the first row of Section II by $a_{22}^{(1)}$ and write down the results in the fourth row of Section II.

(8) Check as in (4).

(9) Compute $a_{ij}^{(2)}$ ($i=3, 4$; $j=3, 4, 5$) by formula (7) of Sec. 3.2. Enter the results in the first two rows of Section III.

(10) Check as in (6).

(11) Divide the elements of the first row of Section III by $a_{33}^{(2)}$ and find the numbers $b_{3j}^{(2)}=a_{3j}^{(2)}/a_{33}^{(2)}$. Write down the results in the third row of Section III.

(12) Make a check.

(13) Compute $a_{4j}^{(3)}=a_{4j}^{(2)}-a_{43}^{(2)}b_{3j}^{(2)}$, and enter the results in Section IV.

Reverse procedure:

(1) Write down the unities in Section V as is indicated in Table 6.

(2) Compute $x_4=\dfrac{a_{45}^{(3)}}{a_{44}^{(3)}}$.

(3) For computing the values x_3, x_2, x_1 use only the rows of Sections I, II, III containing units *(marked rows)*, beginning with the last. Thus, to compute x_3 multiply x_4 by $b_{34}^{(2)}$ and subtract the result from $b_{35}^{(2)}$. The units put in Section V help us to find the corresponding coefficients for x_i ($i=3, 2, 1$) in the marked rows.

Hence,

$$x_3=b_{35}^{(2)}-b_{34}^{(2)}x_4.$$

(4) Compute x_2 for which purpose use the elements of the marked row of Section II:

$$x_2=b_{25}^{(1)}-b_{24}^{(1)}x_4-b_{23}^{(1)}x_3.$$

(5) Compute x_1 for which purpose use the elements of the marked row of Section I:

$$x_1=b_{15}-b_{14}x_4-b_{13}x_3-b_{12}x_2.$$

In the check scheme the reverse procedure is carried out in a similar way. The solutions of this scheme must differ from those of the given scheme by 1 (accurate to one unit of the last retained digit).

$$\bar{x}_i=x_i+1 \quad (i=1, 2, 3, 4).$$

This check is carried out by means of the $\sum$ column. For systems with a different number of unknowns the Gaussian compact scheme is realized in the same way. This scheme turns out

to be especially useful in a simultaneous solving of several systems differing only in the columns of constant terms, which takes place, for instance, in computing the elements of the inversion of a matrix (see Sec. 3.8.).

Example 1. Solve system (12) of Sec. 3.2 using the Gaussian compact scheme.

Solution. Since all computations were carried out in Sec. 3.2, let us consider only the order of filling in the table with a simultaneous check of the computations.

Direct procedure:

(1) Write down the coefficients of the system a_{ij} for $i = 1, 2, 3, 4, 5$; $j = 1, 2, 3, 4, 5, 6$ in the first section of Table 7.

Table 7

The Gaussian Compact Scheme for System (12) of Sec. 3.2

	i	a_{i1}	a_{i2}	a_{i3}	a_{i4}	a_{i5}	$\sum = a_{i6}$
I	1	2.0	1.0	—0.1	1.0	2.7	6.6
	2	0.4	0.5	4.0	—8.5	21.9	18.3
	3	0.3	—1.0	1.0	5.2	—3.9	1.6
	4	1.0	0.2	2.5	—1.0	9.9	12.6
		1	0.50	—0.05	0.50	1.35	3.30
II	2		0.30	4.02	—8.70	21.36	16.98
	3		—1.15	1.015	5.05	—4.305	0.610
	4		—0.30	2.55	—1.50	8.55	9.300
			1	13.40	—29.00	71.20	56.60
III	3			16.425	—28.300	77.575	65.700
	4			6.570	—10.200	29.910	26.280
				1	—1.72298	4.72298	4.00000
IV	4				1.11998	—1.11998	0
V					1	—1.00000	0.00000
				1		3.00000	4.00000
			1			2.00000	3.00000
		1				1.00000	2.00000

(2) Compute the sums of the coefficients along the row. Thus, for $i=1$ we have

$$\sum_{j=1}^{5} a_{1j}=2.0+1.0-0.1+1.0+2.7=6.6;$$

enter the result in the first row of the $\sum$ column; and so on.

(3) Divide the elements of the first row by $a_{11}=2.0$, and write down the results in the fifth row of Section I.

(4) *Checking:* computing the sum of the first five numbers obtained in (3), we get 3.30, which completely coincides with the number obtained in the $\sum$ column.

(5) Find the numbers $a_{ij}^{(1)}$ ($i=2,\ 3,\ 4$; $j=2,\ 3,\ 4,\ 5,\ 6$) and write them down in Section II.

(6) *Checking:* sum the obtained coefficients in each row. Thus, for $i=2$ we have $\sum_{j=2}^{5} a_{2j}^{(1)}=16.98$. The result coincides with the check number.

(7) Divide the elements of the first row of Section II by $a_{22}^{(1)}=0.3$. Write down the result in the last row of the section.

(8) *Checking:* the sum $1+b_{23}^{(1)}+b_{24}^{(1)}+b_{25}^{(1)}=56.60=b_{26}^{(1)}$.

(9) Determine the numbers $a_{ij}^{(2)}$ ($i=3,\ 4$; $j=3,\ 4,\ 5,\ 6$) and write them down in Section III.

(10) *Checking:* for $i=3$ we have $a_{33}^{(2)}+a_{34}^{(2)}+a_{35}^{(2)}=65.700=a_{36}^{(2)}$; for $i=4$ we have $a_{43}^{(2)}+a_{44}^{(2)}+a_{45}^{(2)}=26.280=a_{46}^{(2)}$.

(11) Divide the elements of the first row of Section III by $a_{33}^{(2)}=16.425$. Enter the result in Section IV.

(12) *Checking:* $1+b_{44}^{(2)}+b_{45}^{(2)}=4.00000=b_{46}^{(2)}$.

(13) Compute $a_{4j}^{(3)}$ ($j=4,\ 5,\ 6$). Write down the result in Section IV.

(14) *Checking:* $a_{44}^{(3)}+a_{45}^{(3)}=0=a_{46}^{(3)}$.

Reverse procedure:

Following the sequence of operations indicated in (1) to (5), in carrying out the reverse procedure, we get the values of the unknowns

$$x_4=-1.00000;\quad x_3=3.00000;\quad x_2=2.00000;\quad x_1=1.00000.$$

The solution of the check system:

$$\bar{x}_4=0.00000;\quad \bar{x}_3=4.00000;\quad \bar{x}_2=3.00000,\quad \bar{x}_1=2.00000.$$

The comparison of the obtained solutions shows that there is no error in the computations.

2. The Modification of Crout-Doolittle. All the computations carried out by this scheme are entered in one table (Table 8). For the sake of simplicity, let us consider the order of filling in the modified scheme for a system of equations in four unknowns.

The table is filled in in the following order.

Table 8

The Crout-Doolittle Computational Scheme

	a_{i1}	a_{i2}	a_{i3}	a_{i4}	a_{i5}	$\sum = a_{i6}$
I	a_{11} a_{21} a_{31} a_{41}	a_{12} a_{22} a_{32} a_{42}	a_{13} a_{23} a_{33} a_{43}	a_{14} a_{24} a_{34} a_{44}	a_{15} a_{25} a_{35} a_{45}	$a_{16} = \sum a_{1j}$ $a_{26} = \sum a_{2j}$ $a_{36} = \sum a_{3j}$ $a_{46} = \sum a_{4j}$
II	m_{21} m_{31} m_{41}					
III		$a_{22}^{(1)}$	$a_{23}^{(1)}$	$a_{24}^{(1)}$	$a_{25}^{(1)}$	$a_{26}^{(1)}$
IV		m_{32} m_{42}				
V			$a_{33}^{(2)}$	$a_{34}^{(2)}$	$a_{35}^{(2)}$	$a_{36}^{(2)}$
VI			m_{43}			
VII				$a_{44}^{(3)}$	$a_{45}^{(3)}$	$a_{46}^{(3)}$
VIII	1	1	1	1	x_4 x_3 x_2 x_1	$\bar{x}_4$ $\bar{x}_3$ $\bar{x}_2$ $\bar{x}_1$

Direct procedure:

(1) Write the coefficients of the system a_{ij} for $i = 1, 2, 3, 4$; $j = 1, 2, 3, 4, 5$ in Section I of Table 8.

(2) Sum the coefficients of each row and enter the results in the $\sum$ column as a_{i6} $(i = 1, 2, 3, 4)$.

(3) Find the numbers

$$m_{i1}=\frac{a_{i1}}{a_{11}}$$

for $i=2, 3, 4$ and write them down in Section II.

(4) Compute the coefficients $a_{2j}^{(1)}$ for $j=2, 3, 4, 5, 6$ by the formula $a_{2j}^{(1)}=a_{2j}-m_{21}a_{1j}$, and write them down in Section III.

(5) *Checking:* the sum $\sum_{j=2}^{5} a_{2j}^{(1)}$ should not differ from $a_{26}^{(1)}$ by more than one unit of the last retained digit (if the computations are performed with a constant number of decimal digits).

(6) Find the numbers m_{i2} for $i=3, 4$ by the formula

$$m_{i2}=\frac{(a_{i2}-m_{i1}a_{12})}{a_{22}^{(1)}}$$

and enter them in Section IV.

(7) Compute the coefficients $a_{3j}^{(2)}$ for $j=3, 4, 5, 6$ by the formula

$$a_{3j}^{(2)}=a_{3j}-m_{31}a_{1j}-m_{32}a_{2j}^{(1)}$$

and write them down in Section V.

(8) *Checking:* compare $\sum_{j=3}^{5} a_{3j}^{(2)}$ with the number $a_{36}^{(2)}$.

(9) Find the number

$$m_{43}=(a_{43}-m_{41}a_{13}-m_{42}a_{23}^{(1)})/a_{33}^{(2)}$$

and enter it in Section VI.

(10) Find the coefficients $a_{4j}^{(3)}$ for $j=4, 5, 6$ by the formula

$$a_{4j}^{(3)}=a_{4j}-m_{41}a_{1j}-m_{42}a_{2j}^{(1)}-m_{43}a_{3j}^{(2)}$$

and enter them in Section VII.

(11) *Checking:* compare the sum $a_{44}^{(3)}+a_{45}^{(3)}$ with the number $a_{46}^{(3)}$.

Reverse procedure. We successively find the numbers x_4, x_3, x_2, x_1 by the formulas

$$\begin{aligned}
a_{44}^{(3)}x_4 &= a_{45}^{(3)},\\
a_{33}^{(2)}x_3+a_{34}^{(2)}x_4 &= a_{35}^{(2)},\\
a_{22}^{(1)}x_2+a_{23}^{(1)}x_3+a_{24}^{(1)}x_4 &= a_{25}^{(1)},\\
a_{11}x_1+a_{12}x_2+a_{13}x_3+a_{14}x_4 &= a_{15}.
\end{aligned}$$

Computations by these formulas are carried out without recording the intermediate results. The final results are entered in Section VIII of the table, as also are the solutions of the check system. The checking is as on page 52.

The number of arithmetic operations in both schemes (Tables 6 and 8) is the same, since the same operations are performed though in an altered order, but the records of intermediate com-

putations are considerably reduced which is of great importance when the computations are carried out on key computers.

Problems

Solve the following systems making use of either the Gaussian scheme or the Crout-Doolittle modification scheme. Carry out the computations with five decimal digits.

1. $A=\begin{pmatrix} 3.2 & -1.5 & 0.5 \\ 1.6 & 2.5 & -1.0 \\ 1.0 & 4.1 & -1.5 \end{pmatrix}$, $\boldsymbol{b}=\begin{pmatrix} 0.90 \\ 1.55 \\ 2.08 \end{pmatrix}$.

2. $A=\begin{pmatrix} 1.5 & -0.2 & 0.1 \\ -0.1 & 1.5 & -0.1 \\ -0.3 & 0.2 & -0.5 \end{pmatrix}$, $\boldsymbol{b}=\begin{pmatrix} 0.4 \\ 0.8 \\ 0.2 \end{pmatrix}$.

3. $A=\begin{pmatrix} 2.1 & -4.5 & -2.0 \\ 3.0 & 2.5 & 4.3 \\ -6.0 & 3.5 & 2.5 \end{pmatrix}$, $\boldsymbol{b}=\begin{pmatrix} 19.07 \\ 3.21 \\ -18.25 \end{pmatrix}$.

4. $A=\begin{pmatrix} 3.0 & -2.0 & 5.3 & -2.1 & 1.0 \\ 1.0 & 4.0 & -6.0 & 4.5 & -6.0 \\ 3.0 & 6.0 & -7.3 & -9.0 & 3.4 \\ -2.0 & -3.0 & 1.0 & -4.0 & 6.0 \\ 1.0 & -4.0 & 6.5 & 1.0 & -3.0 \end{pmatrix}$, $\boldsymbol{b}=\begin{pmatrix} 28.3 \\ -36.2 \\ 24.5 \\ 16.2 \\ 4.3 \end{pmatrix}$.

5. $A=\begin{pmatrix} 8.64-\alpha & 1.71 & 5.42 \\ -6.39 & 4.25 & 1.84+\alpha \\ 4.21 & 7.92-\alpha & -3.41 \end{pmatrix}$, $\boldsymbol{b}=\begin{pmatrix} 10.21-\beta \\ 3.41+\beta \\ 12.29 \end{pmatrix}$,

$\alpha=0.5\cdot k$, $k=0, 1, \ldots, 4$, $\beta=0.2\cdot k$, $k=0,1, \ldots, 5$.

6. $A=\begin{pmatrix} 8.30 & 2.62+\alpha & 4.10 & 1.90 \\ 3.92 & 8.45 & 7.78-\alpha & 2.46 \\ 3.77 & 7.21+\alpha & 8.04 & 2.28 \\ 2.21 & 3.65-\alpha & 1.69 & 6.99 \end{pmatrix}$, $\boldsymbol{b}=\begin{pmatrix} -10.65+\beta \\ 12.21 \\ 15.45-\beta \\ -8.35 \end{pmatrix}$,

$\alpha=0.2\cdot k$, $k=0, 1, \ldots, 4$, $\beta=0.2\cdot k$, $k=0, 1, \ldots, 5$.

3.4 THE METHOD OF PRINCIPAL ELEMENTS

Consider a linear system of equations

$$\left.\begin{aligned} a_{11}x_1+a_{12}x_2+\ldots+a_{1n}x_n&=a_{1,\,n+1},\\ a_{21}x_1+a_{22}x_2+\ldots+a_{2n}x_n&=a_{2,\,n+1},\\ \ldots\quad\ldots\quad\ldots\quad\ldots\quad\ldots\quad\ldots&\\ a_{n1}x_1+a_{n2}x_2+\ldots+a_{nn}x_n&=a_{n,\,n+1}. \end{aligned}\right\} \qquad (1)$$

Let us write the augmented rectangular matrix consisting of the coefficients of the system

$$M = \begin{pmatrix} a_{11} \ldots a_{1q} \ldots a_{1n}\, a_{1,\,n+1} \\ \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\ a_{p1} \ldots . a_{pq} \ldots a_{pn}\, a_{p,\,n+1} \\ \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\ a_{n1} \ldots a_{nq} \ldots a_{nn}\, a_{n,\,n+1} \end{pmatrix}.$$

Choose a nonzero (as a rule, the numerically largest) element a_{pq} (of matrix M not belonging to the column of constant terms ($q \neq n+1$)), this element being called *the principal element*. Compute the multipliers

$$m_i = \frac{a_{iq}}{a_{pq}}$$

for all $i \neq p$.

The row of M with index p which contains the principal element is called *the principal row*. Then perform the following operation: from each ith nonprincipal row subtract termwise the principal row multiplied by m_i. We thus obtain a new matrix in which all the elements of the qth column (with the exception of a_{pq}) are equal to zero. Discarding this column and the principal row, we obtain a new matrix M_1 with the number of rows and columns diminished by unity.

Repeat these operations with matrix M_1 to get matrix M_2, and so on. Thus, we obtain a sequence of matrices.

$$M,\ M_1,\ \ldots,\ M_{n-1}$$

the last of which is a two-term row matrix. It is also regarded as the principal row. Then combine into a system all the principal rows beginning with the last. After an appropriate replacement they form a triangular matrix which is equivalent to the initial one. And here the stage of computations, called *the direct procedure*, is terminated. Solving the system with the obtained matrix of coefficients, we find, step by step, the values of the unknowns x_i ($i = 1, 2, \ldots, n$). This stage of computations is called *the reverse procedure*. All the above described computations can be arranged in one table, which will be similar to the Gaussian compact scheme, with a check provided for each stage of computations (see Example 1).

The principal element is chosen so as to make the numbers m_i as small as possible, and thus reduce the error of computations. Therefore, when applying the Gaussian method on a computer, a scheme with the principal element chosen is usually used.

Note. When carrying out computational work on a computer, the choice of the principal element may turn out to be a rather laborious task if the number of equations in a system is large. Therefore, it is the standard practice to take the first row as the principal one, and the numerically largest element of this row at the principal element.

Example 1. Using the Gaussian scheme with the principal element chosen, solve the system

$$1.1161x_1 + 0.1254x_2 + 0.1397x_3 + 0.1490x_4 = 1.5471,$$
$$0.1582x_1 + 1.1675x_2 + 0.1768x_3 + 0.1871x_4 = 1.6471,$$
$$0.1968x_1 + 0.2071x_2 + 1.2168x_3 + 0.2271x_4 = 1.7471,$$
$$0.2368x_1 + 0.2471x_2 + 0.2568x_3 + 1.2671x_4 = 1.8471.$$

Solution. It is convenient to enter the results of all computations in one table (Table 9) in the following order.

Table 9

The Gaussian Scheme with the Principal Element Chosen

	i	m_i	a_{i1}	a_{i2}	a_{i3}	a_{i4}	a_{i5}	$\sum = a_{ic}$
I	1	0.11759	1.11610	0.12540	0.13970	0.14900	1.54710	3.07730
	2	0.14766	0.15820	1.16750	0.17680	0.18710	1.64710	3.33760
	3	0.17923	0.19680	0.20710	1.21680	0.22710	1.74710	3.59490
	4		0.23680	0.24710	0.25680	1.26710	1.84710	3.85490
II	1	0.09353	1.08825	0.09634	0.10950		1.32990	2.62399
	2	0.11862	0.12323	1.13101	0.13888		1.37436	2.76748
	3		0.15436	0.16281	1.17077		1.41604	2.90398
III	1	0.07296	1.07381	0.08111			1.19746	2.35238
	2		0.10492	1.11170			1.20639	2.42301
IV	1		1.06616				1.10944	2.17560
V	1		1				1.04059	2.04059
	2			1			0.98697	1.98697
	3				1		0.93505	1.93505
	4					1	0.88130	1.88130

Direct procedure:

(1) Write down the coefficients of the system in Section I of the table

$$a_{ij} \quad (i = 1, 2, 3, 4;\ j = 1, 2, 3, 4, 5).$$

(2) Enter the sums of the coefficients of each row in the column $\sum = a_{i6}$.

(3) Find the principal element. In the given system it will be the coefficient $a_{44} = 1.26710$ ($p = 4$, $q = 4$); underline it.

(4) Find the numbers m_i ($i = 1, 2, 3$). To this end divide the elements of the column a_{i4} by a_{44} and write down the results in the column m_i Section I:

$$m_1 = \frac{a_{14}}{a_{44}} = \frac{0.14900}{1.26710} = 0.11759; \quad m_2 = \frac{a_{24}}{a_{44}} = \frac{0.18710}{1.26710} = 0.14766;$$

$$m_3 = \frac{a_{34}}{a_{44}} = \frac{0.22710}{1.26710} = 0.17923.$$

(5) Compute the coefficients of the new matrix. From each row i ($i = 1, 2, 3$) subtract the principal row multiplied by the corresponding element m_i.

Thus, for $i = 1$ we have

$$a_{11}^{(1)} = a_{11} - m_1 a_{41} = 1.11610 - 0.11759 \cdot 0.23680 = 1.08825,$$
$$a_{12}^{(1)} = a_{12} - m_1 a_{42} = 0.12540 - 0.11759 \cdot 0.24710 = 0.09634,$$
$$a_{13}^{(1)} = a_{13} - m_1 a_{43} = 0.13970 - 0.11759 \cdot 0.25680 = 0.10950,$$
$$a_{14}^{(1)} = 0,$$
$$a_{15}^{(1)} = a_{15} - m_1 a_{45} = 1.54710 - 0.11759 \cdot 1.84710 = 1.32990,$$
$$a_{16}^{(1)} = a_{16} - m_1 a_{46} = 3.07730 - 0.11759 \cdot 3.85490 = 2.62399.$$

For $i = 2, 3$ proceed in the same way. The results are entered in Section II. Here the principal row is not written, nor the column a_{i4}, since it consists of zeros.

(6) *Checking:* find the sums $\sum_{j=1}^{5} a_{ij}^{(1)}$ and compare them with $a_{i6}^{(1)}$, for instance, $\sum_{j=1}^{5} a_{1j} = 2.62399 = a_{16}^{(1)}$, and so on.

(7) Choose the principal element and underline it. In our case it will be $a_{33}^{(1)} = 1.17077$.

(8) Divide the elements of the column a_{i3} by $a_{33}^{(1)}$ to get the numbers

$$m_1^{(1)} = \frac{a_{13}^{(1)}}{a_{33}^{(1)}} = \frac{0.10950}{1.17077} = 0.09353; \quad m_2^{(1)} = \frac{a_{23}^{(1)}}{a_{33}^{(1)}} = \frac{0.13888}{1.17077} = 0.11862.$$

(9) Compute the coefficients $a_{ij}^{(2)}$. For this purpose from each line i ($i = 1, 2$) subtract the principal row multiplied by the corresponding m_i. Thus, for $i = 1$ we have

$$a_{11}^{(2)} = a_{11}^{(1)} - m_1^{(1)} a_{31}^{(1)} = 1.08825 - 0.09353 \cdot 0.15436 = 1.07381,$$
$$a_{12}^{(2)} = a_{12}^{(1)} - m_1^{(1)} a_{32}^{(1)} = 0.09634 - 0.09353 \cdot 0.16281 = 0.08111,$$
$$a_{13}^{(2)} = a_{13}^{(1)} - m_1^{(1)} a_{33}^{(1)} = 0,$$
$$a_{15}^{(2)} = a_{15}^{(1)} - m_1^{(1)} a_{35}^{(1)} = 1.32990 - 0.09353 \cdot 1.41604 = 1.19746,$$
$$a_{16}^{(2)} = a_{16}^{(1)} - m_1^{(1)} a_{36}^{(1)} = 2.62399 - 0.09353 \cdot 2.90398 = 2.35238.$$

For $i=2$ the computations are performed analogously. The results are entered in Section III, the columns a_{i3} and a_{i4} being left empty.

(10) *Checking:* the sum $\sum_{j=1}^{5} a_{ij}$ $(i=1, 2)$ must be equal to a_{i6}; this condition is met.

(11) Choose the principal element and underline it. Now it will be $a_{22}^{(2)}=1.11170$.

(12) Find $m_1^{(2)}=\frac{a_{12}^{(2)}}{a_{22}^{(2)}}=\frac{0.08111}{1.11170}=0.07296$. Write down the result in the column m_i of Section III.

(13) Subtract the second (principal) row multiplied by $m_1^{(2)}$ from the first row. We get

$$a_{11}^{(3)}=a_{11}^{(2)}-m_1^{(2)}a_{21}^{(2)}=1.07381-0.10492\cdot 0.07296=1.06616,$$
$$a_{12}^{(3)}=a_{12}^{(2)}-m_1^{(2)}a_{22}^{(2)}=0,$$
$$a_{15}^{(3)}=a_{15}^{(2)}-m_1^{(2)}a_{25}^{(2)}=1.19746-0.07296\cdot 1.20639=1.10944,$$
$$a_{16}^{(3)}=a_{16}^{(2)}-m_1^{(2)}a_{26}^{(2)}=2.35238-0.07296\cdot 2.42301=2.17560.$$

Enter the results in Section IV.

(14) *Checking:*

$$a_{11}^{(3)}+a_{15}^{(3)}=1.06616+1.10944=2.17560=a_{16}^{(3)}.$$

(15) Writing out the principal rows of each section, we get a system equivalent to the given one:

$$\left.\begin{aligned}
&1.06616x_1=1.10944,\\
&0.10492x_1+1.11170x_2=1.20639,\\
&0.15436x_1+0.16281x_2+1.17077x_3=1.41604,\\
&0.23680x_1+0.24710x_2+0.25680x_3+1.26710x_4=1.84710.
\end{aligned}\right\} \quad (2)$$

Reverse procedure. The results of computations carried out in realizing the reverse procedure are entered in Section V. We get in succession:

$$x_1=1.10944/1.06616=1.04059,$$
$$x_2=(1.20639-0.10492\cdot 1.04059)/1.11170=0.98697,$$
$$x_3=(1.41604-0.15436\cdot 1.04059-0.16281\cdot 0.98697)\,1.17077=$$
$$=0.93505,$$
$$x_4=(1.84710-0.23680\cdot 1.04059-0.24710\cdot 0.98697-0.25680\times$$
$$\times 0.93505)/1.26710=0.88130.$$

The reverse procedure is checked with the aid of the column $\sum$ in the same way as on page 52.

Problems

1. Solve the following systems using both the Gaussian method with the principal element chosen and the ordinary Gaussian method, carrying out all the computations with five significant digits. Compare the obtained values with the indicated exact values:

(a) $A=\begin{pmatrix}0.15 & 2.11 & 30.75\\ 0.64 & 1.21 & 2.05\\ 3.21 & 1.53 & 1.04\end{pmatrix}$, $\boldsymbol{b}=\begin{pmatrix}-26.38\\ 1.01\\ 5.23\end{pmatrix}$, $\boldsymbol{x}=\begin{pmatrix}1\\ 2\\ -1\end{pmatrix}$,

(b) $A=\begin{pmatrix}1.15 & 0.42 & 100.71\\ 1.19 & 0.55 & 0.32\\ 1.00 & 0.35 & 3.00\end{pmatrix}$, $\boldsymbol{b}=\begin{pmatrix}-198.70\\ 2.29\\ -0.65\end{pmatrix}$, $\boldsymbol{x}=\begin{pmatrix}2\\ 1\\ -2\end{pmatrix}$,

2. Solve the following system using (a) the common Gaussian method; (b) the Gaussian method with the principal element chosen:

$$x+592y=437,$$
$$592x+4308y=2251.$$

Carry out all the computations with four significant digits.

For the following systems find the values of the unknowns with the indicated number m of significant digits using the Gaussian method with the principal element chosen.

3. $A=\begin{pmatrix}2 & 6 & -1\\ 5 & -1 & 2\\ -3 & -4 & 1\end{pmatrix}$, $\boldsymbol{b}=\begin{pmatrix}-12\\ 29\\ 5\end{pmatrix}$, $m=4$.

4. $A=\begin{pmatrix}2.10 & -4.50 & -2.00\\ 3.00 & 2.50 & 4.30\\ -6.00 & 3.50 & 2.50\end{pmatrix}$, $\boldsymbol{b}=\begin{pmatrix}19.07\\ 3.21\\ -18.25\end{pmatrix}$, $m=3$.

5. $A=\begin{pmatrix}21.547 & -95.510 & -96.121\\ 10.223 & -91.065 & -7.343\\ 51.218 & 12.269 & 86.457\end{pmatrix}$, $\boldsymbol{b}=\begin{pmatrix}-49.930\\ -12.465\\ 60.812\end{pmatrix}$, $m=4$.

6. $A=\begin{pmatrix}3.24 & -2.18 & 5.09 & -2.37 & 1.21\\ 0.73 & 3.85 & -6.23 & 4.80 & -5.93\\ 2.88 & 5.73 & -7.02 & -9.17 & 3.58\\ 2.10 & 3.02 & -0.78 & 3.85 & -6.00\\ 1.20 & -4.13 & 6.48 & 0.00 & -3.24\end{pmatrix}$, $\boldsymbol{b}=\begin{pmatrix}28.38\\ -36.00\\ 24.48\\ -16.23\\ 4.34\end{pmatrix}$,

$m=3$.

7. $A=\begin{pmatrix} 2.6 & -4.5 & -2.0 \\ 3.0 & 3.0 & 4.3 \\ -6.0 & 3.5 & 3.0 \end{pmatrix}$, $\boldsymbol{b}=\begin{pmatrix} 19.07 \\ 3.21 \\ -18.25 \end{pmatrix}$, $m=5$.

8. $A=\begin{pmatrix} 4.21 & 22.42+\alpha & 3.85 \\ 2.31 & 31.49 & 1.52 \\ 3.49 & 4.85 & 28.72+\alpha \end{pmatrix}$, $\boldsymbol{b}=\begin{pmatrix} 30.24 \\ 40.95-\beta \\ 42.81 \end{pmatrix}$, $m=5$,

$\alpha=0.25\cdot k$, $k=0, 1, \ldots, 4$, $\beta=0.35\cdot k$, $k=0, 1, \ldots, 5$.

9. $A=\begin{pmatrix} 3.81 & 0.25 & 1.28 & 0.75+\alpha \\ 2.25 & 1.32 & 4.58+\alpha & 0.49 \\ 5.31 & 6.28+\alpha & 0.98 & 1.04 \\ 9.39+\alpha & 2.45 & 3.35 & 2.28 \end{pmatrix}$,

$\boldsymbol{b}=\begin{pmatrix} 4.21 \\ 6.47+\beta \\ 2.38 \\ 10.48+\beta \end{pmatrix}$, $m=5$,

$\alpha=0.5\cdot k$, $k=0, 1, \ldots, 4$, $\beta=0.5\cdot k$, $k=0, 1, \ldots, 5$.

3.5 THE SCHEME OF KHALETSKY

Consider a system of linear equations written in matrix notation as

$$A\boldsymbol{x}=\boldsymbol{b},$$

where $A=(a_{ij})$ is a square matrix $(i, j=1, 2, \ldots, n)$ and

$$\boldsymbol{x}=\begin{pmatrix} x_1 \\ \ldots \\ x_n \end{pmatrix}, \quad \boldsymbol{b}=\begin{pmatrix} a_{1,n+1} \\ \ldots \\ a_{n,n+1} \end{pmatrix}$$

are column vectors.

Represent matrix A in the form of a product $A=BC$, where

$$B=\begin{pmatrix} b_{11} & 0 & 0 & \ldots & 0 \\ b_{21} & b_{22} & 0 & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ b_{n1} & b_{n2} & b_{n3} & \ldots & b_{nn} \end{pmatrix}, \quad C=\begin{pmatrix} 1 & c_{12} & \ldots & c_{1n} \\ 0 & 1 & \ldots & c_{2n} \\ \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & \ldots & 1 \end{pmatrix}.$$

Then the elements b_{ij} and c_{ij} are determined from the formulas

$$\left.\begin{aligned} b_{i1}&=a_{i1}, \\ b_{ij}&=a_{ij}-\sum_{k=1}^{j-1} b_{ik}c_{kj} \quad (i\geqslant j>1) \end{aligned}\right\} \tag{1}$$

and

$$\left.\begin{aligned} c_{1j} &= \frac{a_{1j}}{b_{11}}, \\ c_{ij} &= \frac{1}{b_{ii}}\left(a_{ij} - \sum_{k=1}^{i-1} b_{ik}c_{kj}\right) \quad (1 < i < j). \end{aligned}\right\} \tag{2}$$

Whence the desired vector x may be computed from the chain of equations

$$By = b, \quad Cx = y. \tag{3}$$

Since the matrices B and C are triangular, systems (3) are readily solved, namely:

$$\left.\begin{aligned} y_1 &= a_{1,n+1}/b_{11}, \\ y_i &= \left(a_{i,n+1} - \sum_{k=1}^{i-1} b_{ik}y_k\right)/b_{ii} \quad (i > 1) \end{aligned}\right\} \tag{4}$$

and

$$\left.\begin{aligned} x_n &= y_n, \\ x_i &= y_i - \sum_{k=i+1}^{n} c_{ik}x_k \quad (i < n). \end{aligned}\right\} \tag{5}$$

From formulas (4) it is evident that the numbers y_i are advantageously computed together with the coefficients c_{ij}. This method is known as *the scheme of Khaletsky*. This scheme uses the ordinary type of checking by means of sums.

Khaletsky's scheme is convenient for machine computation since in this case the operations of "accumulation" (1) and (2) may be carried out without recording the intermediate results.

Example 1. Solve the system

$$\begin{aligned} 3x_1 + x_2 - x_3 + 2x_4 &= 6, \\ -5x_1 + x_2 + 3x_3 - 4x_4 &= -12, \\ 2x_1 + x_3 - x_4 &= 1, \\ x_1 - 5x_2 + 3x_3 - 3x_4 &= 3. \end{aligned}$$

Solution. Write down the results of computations in Table 10. It consists of two parts:

(1) the left-hand half which gives the scheme for recording the results of computations;

(2) the right-hand half in which the results of computations are entered according to the indicated scheme.

The table is filled in in the following order:

(1) Write down the matrix of the coefficients of the system, its constant terms and the check sums in Section I of Table 10.

Table 10

The Scheme of Khaletsky

	x_1	x_2	x_3	x_4		Σ	x_1	x_2	x_3	x_4		Σ
I	a_{11}	a_{12}	a_{13}	a_{14}	a_{15}	a_{16}	3	1	—1	2	6	11
	a_{21}	a_{22}	a_{23}	a_{24}	a_{25}	a_{26}	—5	1	3	—4	—12	—17
	a_{31}	a_{32}	a_{33}	a_{34}	a_{35}	a_{36}	2	0	1	—1	1	3
	a_{11}	a_{42}	a_{43}	a_{44}	a_{45}	a_{46}	1	—5	3	—3	3	—1
II	b_{11}	1	c_{12}	c_{13}	c_{14}	c_{15}	c_{16}	3.	1	0.333333	—0.333333	0.666667	2	3.666667
	b_{21}	b_{22}	1	c_{23}	c_{24}	c_{25}	c_{26}	—5	2.666667	1	0.5	—0.25	—0.75	0.5
	b_{31}	b_{32}	b_{33}	1	c_{34}	c_{35}	c_{36}	2	—0.666667	2	1	—1.25	—1.75	—2
	b_{41}	b_{42}	b_{43}	b_{44}	1	c_{45}	c_{46}	1	—5.333333	6	2.5	1	3	4
III						y_1	x_1						2	1
						y_2	x_2						—0.75	—1
						y_3	x_3						—1.75	2
						y_4	x_4						3	3

(2) Transfer the elements of the column x_1 from Section I to Section II since $b_{i1}=a_{i1}$ $(i=1, 2, 3, 4)$.

(3) Compute the elements of the first row of Section II. For this purpose divide all elements of the first row of Section I by the element $a_{11}=b_{11}$, by 3 in our case.

We have:

$$c_{12}=\frac{1}{3}=0.333333, \quad c_{13}=\frac{1}{3}=-0.333333, \quad c_{14}=\frac{2}{3}=0.666667,$$

$$c_{15}=\frac{6}{3}=2, \quad c_{16}=\frac{11}{3}=3.666667.$$

(4) Fill in the column x_2 of Section II beginning with the second row. Using formulas (1), we determine b_{j2}:

$$b_{22}=a_{22}-b_{21}c_{12}=1-\left(-5\cdot\frac{1}{3}\right)=\frac{8}{3}=2.666667,$$

$$b_{32}=a_{32}-b_{31}c_{12}=0-2\cdot\frac{1}{3}=-\frac{2}{3}=-0.666667,$$

$$b_{42}=a_{42}-b_{41}c_{12}=-5-1\cdot\frac{1}{3}=-\frac{16}{3}=-5.333333.$$

(5) Fill in the second row of Section II determining c_{2j} for $j=3, 4, 5, 6$ from formulas (2):

$$c_{23}=\frac{1}{b_{22}}(a_{23}-b_{21}c_{13})=\frac{3}{8}\left(3-5\cdot\frac{1}{3}\right)=\frac{1}{2}=0.5,$$

$$c_{24}=\frac{1}{b_{22}}(a_{24}-b_{21}c_{14})=\frac{3}{8}\left(-4-(-5)\cdot\frac{2}{3}\right)=-\frac{1}{4}=-0.25,$$

$$c_{25}=\frac{1}{b_{22}}(a_{25}-b_{21}c_{15})=\frac{3}{8}(-12-(-5)\cdot 2)=-\frac{3}{4}=-0.75,$$

$$c_{26}=\frac{1}{b_{22}}(a_{26}-b_{21}c_{16})=\frac{3}{8}\left(-17-(-5)\cdot\frac{11}{3}\right)=\frac{1}{2}=0.5.$$

(6) Fill in the column x_3, computing its elements b_{33} and b_{43} by formulas (1).

(7) Proceed analogously until Section II is filled in completely. We thus get a staircase arrangement in Section II: column-row, column-row, etc.

(8) Determine y_i and x_i $(i=1, 2, 3, 4)$, using formulas (4) and (5), and enter them in Section III:

$$y_1=a_{15}/b_{11}=6/3=2,$$

$$y_2=(a_{25}-b_{21}y_1)/b_{22}=(-12+5\cdot 2)/2.666667=-0.75,$$

$$y_3=(a_{35}-b_{31}y_1-b_{32}y_2)/b_{33}=(1-2\cdot 2-0.666667\cdot 0.75)\,2=-1.75,$$

$$y_4=(a_{45}-b_{41}y_1-b_{42}y_2-b_{43}y_3)/b_{44}=$$

$$=(3-2-5.333333\cdot 0.75+6\cdot 1.75)=3,$$

$$x_4=y_4=3,$$

$$x_3=y_3-c_{34}x_4=-1.75+1.25\cdot 3=2,$$

$$x_2 = y_2 - c_{23}x_3 - c_{24}x_4 = -0.75 - 0.5 \cdot 2 + 0.25 \cdot 3 = -1,$$
$$x_1 = y_1 - c_{12}x_2 - c_{13}x_3 - c_{14}x_4 =$$
$$= 2 + 0.333333 + 0.333333 \cdot 2 - 0.666667 \cdot 3 = 1.$$

(9) Intermediate checking is done by means of the $\sum$ column, which is involved in the same operations as is the column of constant terms.

Problems

Solve the following systems, using the scheme of Khaletsky:

1. $A = \begin{pmatrix} 2.5 & -3.0 & 4.6 \\ -3.5 & 2.6 & 1.5 \\ -6.5 & -3.5 & 7.3 \end{pmatrix}$, $\boldsymbol{b} = \begin{pmatrix} -1.05 \\ -14.46 \\ -17.735 \end{pmatrix}$.

2. $A = \begin{pmatrix} 2.0 & -4.0 & -3.25 & 1.0 \\ 3.0 & -3.0 & -4.3 & 8.0 \\ 1.0 & -5.0 & 3.3 & -20.0 \\ 2.5 & -4.0 & 2.0 & -3.0 \end{pmatrix}$, $\boldsymbol{b} = \begin{pmatrix} 4.84 \\ 8.89 \\ -14.01 \\ -20.29 \end{pmatrix}$.

3. $A = \begin{pmatrix} 2 & -1 & 4 & -3 & 1 \\ -1 & 1 & 2 & 1 & 3 \\ 4 & 2 & 3 & 3 & -1 \\ -3 & 1 & 3 & 2 & 4 \\ 1 & 3 & -1 & 4 & 4 \end{pmatrix}$, $\boldsymbol{b} = \begin{pmatrix} 11 \\ 14 \\ 4 \\ 16 \\ 18 \end{pmatrix}$.

3.6 THE SQUARE-ROOT METHOD

The square-root method (see [2], [12], [26], [54]) is used for solving a linear system

$$A\boldsymbol{x} = \boldsymbol{b}, \tag{1}$$

where $A = [a_{ij}]$ is a symmetric matrix, that is

$$a_{ij} = a_{ji} \quad (i, j = 1, 2, \ldots, n).$$

This method is more economical and convenient as compared with the above considered methods of solving systems of the general form.

In practical applications of the square-root method, the direct procedure is used to compute successively the coefficients t_{ij} and y_i $(i = 1, 2, \ldots, n)$ and then the reverse procedure is used to find the unknowns x_i $(i = n, n-1, \ldots, 1)$.

Direct procedure. Let us represent the matrix A in the form of a product of two transposed triangular matrices:

$$A = T'T, \tag{2}$$

where

$$T=\begin{pmatrix} t_{11} & t_{12} & \dots & t_{1n} \\ 0 & t_{22} & \dots & t_{2n} \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & t_{nn} \end{pmatrix}, \quad T'=\begin{pmatrix} t_{11} & 0 & \dots & 0 \\ t_{12} & t_{22} & \dots & 0 \\ \dots & \dots & \dots & \dots \\ t_{1n} & t_{2n} & \dots & t_{nn} \end{pmatrix}.$$

Multiplying together the matrices T' and T and equating the product to the matrix A, we get the following formulas for determining t_{ij}:

$$\left.\begin{aligned} &t_{11}=\sqrt{a_{11}},\ t_{1j}=\frac{a_{1j}}{t_{11}} \quad (j>1), \\ &t_{ii}=\sqrt{a_{ii}-\sum_{k=1}^{i-1} t_{ki}^2} \quad (1<i\leqslant n), \\ &t_{ij}=\frac{a_{ij}-\sum_{k=1}^{i-1} t_{ki}t_{kj}}{t_{ii}} \quad (i<j), \\ &t_{ij}=0 \ \text{ at } \ i>j. \end{aligned}\right\} \tag{3}$$

On finding the matrix T, we replace system (1) by two equivalent systems with triangular matrices

$$T'\boldsymbol{y}=\boldsymbol{b}, \quad T\boldsymbol{x}=\boldsymbol{y}. \tag{4}$$

Reverse procedure. Write systems (4) in the expanded form:

$$\left.\begin{aligned} &t_{11}y_1=b_1, \\ &t_{12}y_1+t_{22}y_2=b_2, \\ &\dots\dots\dots\dots \\ &t_{1n}y_1+t_{2n}y_2+\dots+t_{nn}y_n=b_n, \end{aligned}\right\} \tag{5}$$

$$\left.\begin{aligned} &t_{11}x_1+t_{12}x_2+\dots+t_{1n}x_n=y_1, \\ &t_{22}x_2+\dots+t_{2n}x_n=y_2, \\ &\dots\dots\dots\dots\dots\dots \\ &t_{nn}x_n=y_n. \end{aligned}\right\} \tag{6}$$

Whence we successively find

$$y_1=\frac{b_1}{t_{11}}, \quad y_i=\frac{b_i-\sum_{k=1}^{i-1} t_{ki}y_k}{t_{ii}} \quad (i>1), \tag{7}$$

$$x_n=\frac{y_n}{t_{nn}}, \quad x_i=\frac{y_i-\sum_{k=i+1}^{n} t_{ik}x_k}{t_{ii}} \quad (i<n). \tag{8}$$

The computations are accompanied with an ordinary checking procedure with the aid of sums, all coefficients of the appropriate row being taken into consideration when forming a sum (see Example 1).

Note that with real a_{ij} we can obtain pure imaginary t_{ij}. In this case the method is applicable as well (see Example 2).

The square-root method saves much time as compared with the above considered methods since, firstly, it essentially reduces the number of multiplications and divisions (to about half for large n, see [2]), secondly, it permits accumulating a sum of products without recording the intermediate results.

Example 1. Using the square-root method, solve the following system of equations:

$$\left.\begin{aligned} 1.00x_1+0.42x_2+0.54x_3+0.66x_4&=0.3,\\ 0.42x_1+1.00x_2+0.32x_3+0.44x_4&=0.5,\\ 0.54x_1+0.32x_2+1.00x_3+0.22x_4&=0.7,\\ 0.66x_1+0.44x_2+0.22x_3+1.00x_4&=0.9. \end{aligned}\right\} \tag{9}$$

Solution. Direct procedure:

(1) Write down the coefficients of the system in Section I of Table 11.

(2) Summate the coefficients moving from row to row and enter the results in the last column as the elements S_1, S_2, S_3, S_4.

(3) Find t_{ij}. For this purpose with the aid of the general formulas (3) write the formulas for computing t_{ij} for $n=4$:

$$\left.\begin{aligned} &t_{11}=\sqrt{a_{11}},\quad t_{12}=\frac{a_{12}}{\sqrt{a_{11}}},\quad t_{13}=\frac{a_{13}}{\sqrt{a_{11}}},\quad t_{14}=\frac{a_{14}}{\sqrt{a_{11}}},\\ &t_{22}=\sqrt{a_{22}-t_{12}^2},\quad t_{23}=\frac{a_{23}-t_{12}t_{13}}{t_{22}},\quad t_{24}=\frac{a_{24}-t_{12}t_{14}}{t_{22}},\\ &t_{33}=\sqrt{a_{33}-t_{13}^2-t_{23}^2},\quad t_{34}=\frac{a_{34}-t_{13}t_{14}-t_{23}t_{24}}{t_{33}},\\ &t_{44}=\sqrt{a_{44}-t_{14}^2-t_{24}^2-t_{34}^2}. \end{aligned}\right\} \tag{10}$$

Using these formulas, we successively obtain

$$t_{11}=1.00,\quad t_{12}=0.42,\quad t_{13}=0.54,\quad t_{14}=0.66,$$

$$t_{22}=\sqrt{1.00-0.42^2}=\sqrt{1.42\cdot 0.58}=0.90752,$$

$$t_{23}=\frac{0.32-0.42\cdot 0.54}{0.90752}=0.10270,\quad t_{24}=\frac{0.44-0.42\cdot 0.66}{0.90752}=0.17939$$

and so on.

Write down the results in Section II according to the scheme given in the left-hand part of Table 11.

Table 11

The Square-Root Method for System (9)

I	a_{11}	a_{12}	a_{13}	a_{14}	b_1	S_1	1.00	0.42	0.54	0.66	0.30	2.92
		a_{22}	a_{23}	a_{24}	b_2	S_2	0.42	1.00	0.32	0.44	0.50	2.68
			a_{33}	a_{34}	b_3	S_3	0.54	0.32	1.00	0.22	0.70	2.78
				a_{44}	b_4	S_4	0.66	0.44	0.22	1.00	0.90	3.22
II	t_{11}	t_{12}	t_{13}	t_{14}	y_1	σ_1	1.00	0.42	0.54	0.66	0.30	2.92
		t_{22}	t_{23}	t_{24}	y_2	σ_2		0.90752	0.10270	0.17939	0.41211	1.60173
			t_{33}	t_{34}	y_3	σ_3			0.83537	−0.18533	0.59336	1.24340
				t_{44}	y_4	σ_4				0.70560	1.04597	1.75157
III	x_1	x_2	x_3	x_4			−1.25778	0.04348	1.03917	1.48238		
	$\bar{x}_1$	$\bar{x}_2$	$\bar{x}_3$	$\bar{x}_4$			−0.25779	1.04349	2.03917	2.48238		

(4) Compute the elements σ_1, σ_2, σ_3, σ_4 by formulas analogous to formulas (10):

$$\sigma_1 = \frac{S_1}{t_{11}} = \frac{2.92}{1.00} = 2.92, \quad \sigma_2 = \frac{S_2 - a_{12}\sigma_1}{t_{22}} = \frac{2.68 - 0.42 \cdot 2.92}{0.90752} = 1.60173,$$

$$\sigma_3 = \frac{S_3 - a_{13}\sigma_1 - a_{23}\sigma_2}{t_{33}} = \frac{2.78 - 0.54 \cdot 2.92 - 0.32 \cdot 1.60173}{0.83537} = 1.24340,$$

$$\sigma_4 = \frac{S_4 - a_{14}\sigma_1 - a_{24}\sigma_2 - a_{34}\sigma_3}{t_{44}} = 1.75157.$$

Reverse procedure.

(5) Find y_i $(i = 1, 2, 3, 4)$. By formulas (7) we shall successively have

$$y_1 = \frac{b_1}{t_{11}} = \frac{0.3}{1.00} = 0.30, \quad y_2 = \frac{b_2 - t_{12}y_1}{t_{22}} = \frac{0.5 - 0.42 \cdot 0.3}{0.90752} = 0.41211,$$

$$y_3 = \frac{b_3 - t_{13}y_1 - t_{23}y_2}{t_{33}} = \frac{0.7 - 0.54 \cdot 0.3 - 0.10270 \cdot 0.41211}{0.83537} = 0.59336,$$

$$y_4 = \frac{b_4 - t_{14}y_1 - t_{24}y_2 - t_{34}y_3}{t_{44}} = 1.04597.$$

The values of y_i are entered in Section II.

(6) *Checking*: the sum of elements of a row in the first five columns must be equal to the appropriate element of the last column. Find these sums:

$$\begin{aligned} t_{11} + t_{12} + t_{13} + t_{14} + y_1 &= 2.92, \\ t_{22} + t_{23} + t_{24} + y_2 &= 1.60173, \\ t_{33} + t_{34} + y_3 &= 1.24340, \\ t_{44} + y_4 &= 1.75157. \end{aligned}$$

Thus, the results coincide with the values of the elements of the last column. Now we may pass over to the next stage of computations.

(7) Find x_i $(i = 1, 2, 3, 4)$. On the basis of formulas (8) we get for $n = 4$

$$\left.\begin{aligned} x_4 &= \frac{y_4}{t_{44}} = \frac{1.04597}{0.70560} = 1.48238, \\ x_3 &= \frac{y_3 - t_{34}x_4}{t_{33}} = 1.03917, \\ x_2 &= \frac{y_2 - t_{23}x_3 - t_{24}x_4}{t_{22}} = 0.04348, \\ x_1 &= \frac{y_1 - t_{12}x_2 - t_{13}x_3 - t_{14}x_4}{t_{11}} = -1.25778. \end{aligned}\right\} \qquad (11)$$

The results are entered in Section III.

(8) *Checking*: the numbers x_i are found by formulas (11), replacing the elements y_i by σ_i $(i = 1, 2, 3, 4)$. If all the computations

are carried out correctly, it must be

$$\bar{x}_i = x_i + 1.$$

Comparison shows that the quantities $\bar{x}_i$ and $x_i + 1$ coincide for $i = 3, 4$, while for $i = 1, 2$ they differ by only unity of the fifth decimal place which is quite permissible.

Example 2. Using the square-root method, solve the following system of equations

$$\left.\begin{aligned} x_1 + 3x_2 - 2x_3 \qquad - 2x_5 &= 0.5, \\ 3x_1 + 4x_2 - 5x_3 + x_4 - 3x_5 &= 5.4, \\ -2x_1 - 5x_2 + 3x_3 - 2x_4 + 2x_5 &= 5.0, \\ x_2 - 2x_3 + 5x_4 + 3x_5 &= 7.5, \\ -2x_1 - 3x_2 + 2x_3 + 3x_4 + 4x_5 &= 3.3. \end{aligned}\right\} \qquad (12)$$

Solution. Write down the coefficients a_{ij} and the constant terms b_i of the given system in Section I of Table 12 and fill in the last column. Using formulas (3) and (7) and moving from row to row in succession, compute the coefficients t_{ij} and the new constant terms y_i, thus filling Section II of the table.

Some values of t_{ij} turn out to be imaginary, for instance:

$$t_{22} = \sqrt{a_{22} - t_{12}^2} = \sqrt{4 - 3^2} = i\sqrt{5} = 2.2361i,$$

$$t_{23} = \frac{a_{23} - t_{12}t_{13}}{t_{22}} = \frac{-5 - 3(-2)}{2.2361i} = -0.4472i.$$

For the purpose of checking fill in the last column. Using formulas (8), we find the values of the unknowns x_i, for instance:

$$x_3 = \frac{y_3 - t_{35}x_5 - t_{34}x_4}{t_{33}} = \frac{-7.5803i - 1.5652i \cdot 0.1998 - 2.0125i \cdot (-0.8996)}{0.8944i} = -6.8011.$$

The check numbers $\bar{x}_i$ are obtained from system (6) by replacing y_i by σ_i. The quantities x_i and $\bar{x}_i$ are entered in Section III. If the computations are performed correctly, then the values $\bar{x}_i$ and $x_i + 1$ may differ only by the last digit. Comparison shows that this condition is met.

Problems

Using the square-root method, solve the following systems of equations. The computations should be carried out with five de-

Table 12

The Square-Root Method for System (12)

I	a_{11}	a_{12}	a_{13}	a_{14}	a_{15}	b_1	S_1	1	3	−2	0	−2	0.5	0.5
		a_{22}	a_{23}	a_{24}	a_{25}	b_2	S_2	3	4	−5	1	−3	5.4	5.4
			a_{33}	a_{34}	a_{35}	b_3	S_3	−2	−5	3	−2	2	5.0	1.0
				a_{44}	a_{45}	b_4	S_4	0	1	−2	5	3	7.5	14.5
					a_{55}	b_5	S_5	−2	−3	2	3	4	3.3	7.3
II	t_{11}	t_{12}	t_{13}	t_{14}	t_{15}	y_1	σ_1	1	3	−2	0	−2	0.5	0.5
		t_{22}	t_{23}	t_{24}	t_{25}	y_2	σ_2		2.2361i	−0.4472i	−0.4472i	−1.3416i	−1.7471i	−1.7471i
			t_{33}	t_{34}	t_{35}	y_3	σ_3			0.8944i	2.0125i	1.5653i	−7.5803i	−3.1081i
				t_{44}	t_{45}	y_4	σ_4				3.0414	2.2194	−2.2928	2.9679
					t_{55}	y_5	σ_5					0.8221i	0.1643i	0.9859i
III	x_1	x_2	x_3	x_4	x_5			−6.0978	−2.2016	−6.8011	−0.8996	0.1998		
	$\bar{x}_1$	$\bar{x}_2$	$\bar{x}_3$	$\bar{x}_4$	$\bar{x}_5$			−5.0973	−1.2017	−5.8004	0.1007	1.1998		

cimal digits.

1. $A=\begin{pmatrix}3.1 & 1.5 & 1.0\\ 1.5 & 2.5 & 0.5\\ 1.0 & 0.5 & 4.2\end{pmatrix}$, $\boldsymbol{b}=\begin{pmatrix}10.83\\ 9.20\\ 17.10\end{pmatrix}$.

2. $A=\begin{pmatrix}3.2 & 1 & 1\\ 1 & 3.7 & 1\\ 1 & 1 & 4.2\end{pmatrix}$, $\boldsymbol{b}=\begin{pmatrix}4\\ 4.5\\ 5\end{pmatrix}$.

3. $A=\begin{pmatrix}2.12 & 0.42 & 1.34 & 0.88\\ 0.42 & 3.95 & 1.87 & 0.43\\ 1.34 & 1.87 & 2.98 & 0.46\\ 0.88 & 0.43 & 0.46 & 4.44\end{pmatrix}$, $\boldsymbol{b}=\begin{pmatrix}11.172\\ 0.115\\ 9.009\\ 9.349\end{pmatrix}$.

4. $A=\begin{pmatrix}5.5 & 7 & 6 & 5.5\\ 7 & 10.5 & 8 & 7\\ 6 & 8 & 10.5 & 9\\ 5.5 & 7 & 9 & 10.5\end{pmatrix}$, $\boldsymbol{b}=\begin{pmatrix}23\\ 32\\ 33\\ 31\end{pmatrix}$.

5. $A=\begin{pmatrix}5.18+\alpha & 1.12 & 0.95 & 1.32 & 0.83\\ 1.12 & 4.28-\alpha & 2.12 & 0.57 & 0.91\\ 0.95 & 2.12 & 6.13+\alpha & 1.29 & 1.57\\ 1.32 & 0.57 & 1.29 & 4.57-\alpha & 1.25\\ 0.83 & 0.91 & 1.57 & 1.25 & 5.21+\alpha\end{pmatrix}$,

$\boldsymbol{b}=\begin{pmatrix}6.19+\beta\\ 3.21\\ 4.28-\beta\\ 6.25\\ 4.95+\beta\end{pmatrix}$, $\alpha=0.25\cdot k$, $k=0, 1, \ldots 4$,
$\beta=0.35\cdot k$, $k=0, 1, 2, \ldots, 5$.

3.7 COMPUTING DETERMINANTS

1. The use of the Gaussian method. It is proved (see [2], [12], [54]), that the determinant of matrix A is equal to the product of the leading (or principal) elements of the corresponding Gaussian scheme, i.e.

$$\Delta=\det A=a_{11}a_{22}^{(1)}\ldots a_{nn}^{(n-1)}. \tag{1}$$

Thus, to evaluate the determinant $\det A$ we have to carry out the computations required for performing the direct procedure in the Gaussian method for a system

$$A\boldsymbol{x}=0,$$

and then find the product of the leading elements. In this case

we use the same computational scheme as was used for solving a system of linear equations (here the column of constant terms in superfluous). The check relations also remain unchanged.

Example 1. Evaluate the determinant

$$\Delta = \begin{vmatrix} 2.0 & 1.0 & -0.1 & 1.0 \\ 0.4 & 0.5 & 4.0 & -8.5 \\ 0.3 & -1.0 & 1.0 & 5.2 \\ 1.0 & 0.2 & 2.5 & -1.0 \end{vmatrix}.$$

Solution. The given determinant coincides with the determinant of the system considered in Example 1 of Sec. 2.3. Using the scheme of this example, we find the product of the leading elements and get the required value of the determinant

$$\Delta = 2.0 \cdot 0.30 \cdot 16.425 \cdot (-1.72298) = -16.99997.$$

Example 2. Evaluate the determinant

$$\Delta = \begin{vmatrix} 1.1161 & 0.1254 & 0.1397 & 0.1490 \\ 0.1582 & 1.1675 & 0.1768 & 0.1871 \\ 0.1968 & 0.2071 & 1.2168 & 0.2271 \\ 0.2368 & 0.2471 & 0.2568 & 1.2671 \end{vmatrix}.$$

Solution. The given determinant is equal to the determinant of the system solved in Example 1 of Sec. 3.4 by the Gaussian method with the principal element chosen. Forming the product of the leading elements, we get the required value of the determinant

$$\Delta = a_{44} \cdot a_{33}^{(1)} \cdot a_{22}^{(2)} \cdot a_{11}^{(3)} = 1.26710 \cdot 1.17077 \cdot 1.11170 \cdot 1.06616 = 1.75829.$$

Example 3. Evaluate the determinant

$$\Delta = \begin{vmatrix} 2.8 & 2.1 & -1.3 & 0.3 \\ -1.4 & 4.5 & -7.7 & 1.3 \\ 0.6 & 2.1 & -5.8 & 2.4 \\ 3.5 & -6.5 & 3.2 & -7.9 \end{vmatrix}.$$

Solution. Let us use the Gaussian compact scheme. Write down the elements of the determinant in Section I of Table 13 and carry out all the computations required for forming the Gaussian compact scheme. The results of these computations are given in Sections II, III, IV of Table 13, with the leading elements underlined. The column $\sum$ is used to check the computations as in solving systems of equations.

Table 13

Evaluating a Determinant by the Gaussian Method

	a_{i1}	a_{i2}	a_{i3}	a_{i4}	Σ
I	2.8	2.1	—1.3	0.3	3.9
	—1.4	4.5	—7.7	1.3	—3.3
	0.6	2.1	—5.8	2.4	—0.7
	3.5	—6.5	3.2	—7.9	—7.7
	1	0.7500	—0.4643	0.1071	1.3928
II		5.5500	—8.3500	1.4499	—1.3501
		1.6500	—5.5214	2.3357	—1.5357
		—9.1250	4.8250	—8.2748	—12.5748
		1	—1.5045	0.2612	—0.2433
III			—3.0390	1.9047	—1.1343
			—8.9036	—5.8913	—14.7949
			1	—0.6268	0.3732
IV				—11.4721	—10.4721

Multiplying together the leading elements of each section, we get

$$\Delta = 2.8 \cdot 5.5500 \cdot (-3.0390) \cdot (-11.4721) = 541.78.$$

2. The use of the square-root method. If A is a symmetric matrix, then it is advisable to use the square-root method for evaluating the determinant of this matrix.

Let us represent matrix A in the form of a product of two mutually transposed triangular matrices:

$$A = T'T,$$

where

$$T = \begin{pmatrix} t_{11} & t_{12} & \dots & t_{1n} \\ 0 & t_{22} & \dots & t_{2n} \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & t_{nn} \end{pmatrix};$$

then

$$\det A = \det T' \det T = (\det T)^2 = (t_{11}t_{22}\ldots t_{nn})^2. \qquad (2)$$

The numbers t_{ii} $(i = 1, 2, \ldots, n)$ are found from the formulas (3) of Sec. 3.6.

Example 4. Evaluate the determinant

$$\Delta = \begin{vmatrix} 1.00 & 0.42 & 0.54 & 0.66 \\ 0.42 & 1.00 & 0.32 & 0.44 \\ 0.54 & 0.32 & 1.00 & 0.22 \\ 0.66 & 0.44 & 0.22 & 1.00 \end{vmatrix}.$$

Solution. In Example 1 (see Sec. 3.6) consideration was given to a system whose determinant coincides with the given one. The system was solved by the square-root method, and the appropriate triangular matrix T was formed. Taking the diagonal elements of this matrix t_{ii} $(i = 1, 2, 3, 4)$, we find by formula (2) (see Table 11)

$$\Delta = (1.00 \cdot 0.90752 \cdot 0.83537 \cdot 0.70560)^2 = (0.53492)^2 = 0.28614.$$

Problems

Using the Gaussian or the square-root methods, evaluate the following determinants.

1. $\Delta = \begin{vmatrix} 8.64-\alpha & 1.71 & 5.42 \\ -6.39 & 4.25 & 1.84+\alpha \\ 4.21 & 7.92-\alpha & -3.41 \end{vmatrix}$, $\alpha = 0.5 \cdot n$,

$n = 0, 1, 2, \ldots, 10.$

2. $\Delta = \begin{vmatrix} 3.81 & 0.25 & 1.28 & 0.75+\alpha \\ 2.25 & 1.32 & 4.58+\alpha & 0.49 \\ 5.31 & 6.28+\alpha & 0.98 & 1.04 \\ 9.39+\alpha & 2.45 & 3.35 & 2.28 \end{vmatrix}$, $\alpha = 0.5 \cdot n$,

$n = 0, 1, 2, \ldots, 10.$

3. $\Delta = \begin{vmatrix} 5.18+\alpha & 1.12 & 0.95 & 1.32 & 0.83 \\ 1.12 & 4.28-\alpha & 2.12 & 0.57 & 0.91 \\ 0.95 & 2.12 & 6.13+\alpha & 1.29 & 1.57 \\ 1.32 & 0.57 & 1.29 & 4.57-\alpha & 1.25 \\ 0.83 & 0.91 & 1.57 & 1.25 & 5.21+\alpha \end{vmatrix}$,

$\alpha = 0.25 \cdot k, \quad k = 0, 1, 2, \ldots, 10.$

3.8 COMPUTING THE ELEMENTS OF AN INVERSE MATRIX BY THE GAUSSIAN METHOD

The inverse of a given matrix A is such a matrix A^{-1}, for which

$$AA^{-1} = A^{-1}A = E,$$

where E is the unit matrix:

$$E = \begin{pmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & & 0 \\ \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & & 1 \end{pmatrix}.$$

A square matrix A is called *nonsingular* if its determinant $\det A$ is nonzero. Every nonsingular matrix has an inverse.

Suppose we have a nonsingular matrix

$$A = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \cdot & \cdot & \cdot & \cdot \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{pmatrix}.$$

To compute the elements of its inverse

$$A^{-1} = \begin{pmatrix} x_{11} & x_{12} & \dots & x_{1n} \\ x_{21} & x_{22} & \dots & x_{2n} \\ \cdot & \cdot & \cdot & \cdot \\ x_{n1} & x_{n2} & \dots & x_{nn} \end{pmatrix}$$

we use the relation

$$AA^{-1} = E.$$

Multiplying matrices A and A^{-1}, and equating each element of the product to the corresponding element of matrix E, we get a system of n^2 equations in n^2 unknowns x_{ij} $(i, j = 1, 2, \dots, n)$.

Thus, multiplying termwise each row of matrix A by the first column of matrix A^{-1} and equating each time the obtained product to the corresponding element of the first column of matrix E, we get the system

$$\left.\begin{aligned} a_{11}x_{11} + a_{12}x_{21} + a_{13}x_{31} + \dots + a_{1n}x_{n1} &= 1, \\ a_{21}x_{11} + a_{22}x_{21} + a_{23}x_{31} + \dots + a_{2n}x_{n1} &= 0, \\ \cdot\ \cdot\ \cdot

Analogously, by termwise multiplying the rows of matrix A by the second column of matrix A^{-1} we form another system

$$\left.\begin{aligned} &a_{11}x_{12}+a_{12}x_{22}+a_{13}x_{32}+\ldots+a_{1n}x_{n2}=0,\\ &a_{21}x_{12}+a_{22}x_{22}+a_{23}x_{32}+\ldots+a_{2n}x_{n2}=1,\\ &\ldots\ldots\ldots\ldots\ldots\ldots\ldots\\ &a_{n1}x_{12}+a_{n2}x_{22}+a_{n3}x_{32}+\ldots+a_{nn}x_{n2}=0, \end{aligned}\right\}\ \text{and so on.} \tag{2}$$

Thus, the system of n^2 equations in n^2 unknowns decomposes into n systems of equations in n unknowns. All these systems have one and the same matrix A and differ only in constant terms. Since in solving a system by the Gaussian method most of the computations are performed on the matrix of the coefficients, we may combine the solutions of all these systems into one scheme, considering simultaneously n columns of constant terms.

All the results of the computations can be arranged in one table. Table 14 gives a computational scheme for a matrix of the fourth order.

Note. An inverse matrix can also be computed by the Gaussian method with the principal element chosen.

Example 1. Find the inverse matrix A^{-1} for the matrix

$$A=\begin{pmatrix} 1.8 & -3.8 & 0.7 & -3.7\\ 0.7 & 2.1 & -2.6 & -2.8\\ 7.3 & 8.1 & 1.7 & -4.9\\ 1.9 & -4.3 & -4.9 & -4.7 \end{pmatrix}. \tag{3}$$

Solution. The computations are given in Table 15. The last column of the table consists of the sums of elements for each row and serves for the checking purpose.

Note that the elements of the rows of the inverse matrix are obtained in reverse order.

Thus, the required inverse matrix has the form

$$A^{-1}=\begin{pmatrix} -0.21121 & -0.46003 & 0.16248 & 0.26956\\ -0.03533 & 0.16873 & 0.01573 & -0.08920\\ 0.23030 & 0.04607 & -0.00944 & -0.19885\\ -0.29316 & -0.38837 & 0.06128 & 0.18513 \end{pmatrix}.$$

To check, form the product

$$AA^{-1}=\begin{pmatrix} 0.99997 & 0.00000 & -0.00001 & 0.00000\\ -0.00025 & 0.99997 & -0.00002 & -0.00039\\ -0.00808 & -0.01017 & 0.99982 & 0.00009\\ 0.00000 & 0.00000 & 0.00000 & 1.00048 \end{pmatrix}.$$

Table 14

Computing an Inverse Matrix by the Gaussian Method

a_{i1}	a_{i2}	a_{i3}	a_{i4}	$a_{i5,\mathrm{I}}$	$a_{i5,\mathrm{II}}$	$a_{i5,\mathrm{III}}$	$a_{i5,\mathrm{IV}}$
a_{11}	a_{12}	a_{13}	a_{14}	1	0	0	0
a_{21}	a_{22}	a_{23}	a_{24}	0	1	0	0
a_{31}	a_{32}	a_{33}	a_{34}	0	0	1	0
a_{41}	a_{42}	a_{43}	a_{44}	0	0	0	1
1	b_{12}	b_{13}	b_{14}	b_{15}	0	0	0
	$a_{22}^{(1)}$	$a_{23}^{(1)}$	$a_{24}^{(1)}$	$a_{25}^{(1)}$, I	1	0	0
	$a_{32}^{(1)}$	$a_{33}^{(1)}$	$a_{34}^{(1)}$	$a_{35}^{(1)}$, I	0	1	0
	$a_{42}^{(1)}$	$a_{43}^{(1)}$	$a_{44}^{(1)}$	$a_{45}^{(1)}$, I	0	0	1
	1	b_{23}	b_{24}	b_{25}, I	b_{25}, II	0	0
		$a_{33}^{(2)}$	$a_{34}^{(2)}$	$a_{35}^{(2)}$, I	$a_{35}^{(2)}$, II	1	0
		$a_{43}^{(2)}$	$a_{44}^{(2)}$	$a_{45}^{(2)}$, I	$a_{45}^{(2)}$, II	0	1
		1	b_{34}	b_{35}, I	b_{35}, II	b_{35}, III	0
			$a_{44}^{(3)}$	$a_{45}^{(3)}$, I	$a_{45}^{(3)}$, II	$a_{45}^{(3)}$, III	1
			1	b_{15}, I	b_{45}, II	b_{45}, III	b_{45}, IV
				x_{41}	x_{42}	x_{43}	x_{44}
				x_{31}	x_{32}	x_{33}	x_{34}
				x_{21}	x_{22}	x_{23}	x_{24}
				x_{11}	x_{12}	x_{13}	x_{14}

The elements of the inverse matrix are obtained with some error which is due to rounding resorted to in the process of computations.

Below we give (see Sec. 3.11) a method for correcting the elements of an approximate inverse matrix.

Problems

Using the Gaussian method, find the inverse of a given matrix

1. $\begin{pmatrix} 8.301 & 2.625 & 4.100 & 1.903 \\ 3.926 & 8.458 & 7.787 & 2.460 \\ 3.773 & 7.211 & 8.041 & 2.280 \\ 2.211 & 3.657 & 1.697 & 6.993 \end{pmatrix}$.

Table 15

Computing the Inverse of Matrix (3)

a_{i1}	a_{i2}	a_{i3}	a_{i4}	$a_{i5,I}$	$a_{i5,II}$	$a_{i5,III}$	$a_{i5,IV}$	Σ
1.8	—3.8	0.7	—3.7	1	0	0	0	—4.0
0.7	2.1	—2.6	—2.8	0	1	0	0	—1.6
7.3	8.1	1.7	—4.9	0	0	1	0	13.2
1.9	—4.3	—4.9	—4.7	0	0	0	1	—11.0
1	—2.11111	0.38889	—2.05556	0.55556	0	0	0	—2.22223
	3 57778	—2.87222	—1.36111	—0.38885	1	0	0	—0.04440
	23.51110	—1.13890	10.10559	—4.05551	0	1	0	29.42228
	—0.28889	—5.63889	—0.79444	—1.05554	0	0	1	—6.77776
	1	—0.80279	—0.38043	—0.10868	0.27950	0	0	—0.01241
		17.73577	19.04992	—1.50032	—6.57135	1	0	29.71405
		—5.87081	—0.90434	—1.08694	0.08074	0	1	—6.78134
		1	1.07411	—0.08459	—0.37108	0.05638	0	1.67539
			5.40155	—1.58355	—2.09780	0.33100	1	3.05456
			1	—0.29316	—0.38837	0.06128	0.18513	0.56540
				0.23030	0.04607	—0.00944	—0.19885	1.06809
				—0.03533	0.16873	0.01573	—0.08920	1.06013
				—0.21121	—0.46003	0.16284	0.26956	0.76266

2. $\begin{pmatrix} 8.644 & 1.715 & 5.422 & 1.433 \\ 2.966 & 6.391 & 1.377 & 1.616 \\ 2.908 & 6.266 & 15.070 & 1.587 \\ 2.522 & 1.535 & 1.331 & 6.188 \end{pmatrix}$.

3. $\begin{pmatrix} 1.1161 & 0.1254 & 0.1397 & 0.1490 \\ 0.1582 & 1.1675 & 0.1768 & 0.1871 \\ 0.1968 & 0.2071 & 1.2168 & 0.2271 \\ 0.2368 & 0.2471 & 0.2568 & 1.2671 \end{pmatrix}$.

4. $\begin{pmatrix} 3.81 & 0.25 & 1.28 & 0.75+\alpha \\ 2.25 & 1.32 & 4.58+\alpha & 0.49 \\ 5.31 & 6.28+\alpha & 0.98 & 1.04 \\ 9.39+\alpha & 2.45 & 3.35 & 2.28 \end{pmatrix}$,

$\alpha = 0{,}5 \cdot n$, $n = 0, 1, \ldots, 4$.

5. $\begin{pmatrix} 24.21+\alpha & 2.42 & 3.85 \\ 2.31 & 31.49 & 1.52 \\ 3.49 & 4.85 & 28.72+\alpha \end{pmatrix}$,

$\alpha = 0.2 \cdot n$, $n = 0, 1, \ldots. 4$.

3.9 THE METHOD OF SIMPLE ITERATION

Let a system of linear equations

$$Ax = b \tag{1}$$

be reduced to the form

$$x = Cx + f, \tag{2}$$

where C is some matrix, and f is a column vector.

Proceeding from an arbitrary vector $x^{(0)}$,

$$x^{(0)} = \begin{pmatrix} x_1^{(0)} \\ x_2^{(0)} \\ \ldots \\ x_n^{(0)} \end{pmatrix},$$

we write out an iteration process

$x^{(k+1)} = Cx^{(k)} + f$ $(k = 0, 1, 2, \ldots)$,

or in full

$$\left.\begin{aligned} x_1^{(k+1)} &= c_{11}x_1^{(k)} + c_{12}x_2^{(k)} + \ldots + c_{1n}x_n^{(k)} + f_1, \\ &\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ x_n^{(k+1)} &= c_{n1}x_1^{(k)} + c_{n2}x_2^{(k)} + \ldots + c_{nn}x_n^{(k)} + f_n. \end{aligned}\right\} \tag{3}$$

Using the iteration process, we get a sequence of vectors $x^{(1)}$, $x^{(2)}$, ..., $x^{(k)}$, ...

It has been proved (see [2], [12], [26], [54], [56]), that if the elements of matrix C satisfy one of the conditions

$$\sum_{j=1}^{n} |c_{ij}| \leqslant \alpha < 1 \quad (i = 1, 2, \ldots, n) \tag{4}$$

or

$$\sum_{i=1}^{n} |c_{ij}| \leqslant \beta < 1 \quad (j = 1, 2, \ldots, n), \tag{5}$$

then the iteration process converges to an exact solution of the system x for any initial vector $x^{(0)}$, i.e.

$$x = \lim_{k \to \infty} x^{(k)}.$$

Thus, the exact solution of the system is obtained only as a result of an infinite process and any vector $x^{(k)}$ of the obtained sequence is an approximate solution. The error of this approximate solution is estimated by one of the following formulas:

$$|x_i - x_i^{(k)}| \leqslant \frac{\alpha}{1-\alpha} \max_{j=1, 2, \ldots, n} |x_j^{(k)} - x_j^{(k-1)}|, \tag{4'}$$

provided condition (4) is fulfilled, or

$$|x_i - x_i^{(k)}| \leqslant \frac{\beta}{1-\beta} \sum_{j=1}^{n} |x_j^{(k)} - x_j^{(k-1)}|, \tag{5'}$$

if condition (5) is satisfied. These estimates can be made still more accurate in the following way:

$$\max |x_i - x_i^{(k)}| \leqslant \frac{\alpha}{1-\alpha} \max |x_i^{(k)} - x_i^{(k-1)}| \tag{4''}$$

and

$$\sum_{i=1}^{n} |x_i - x_i^{(k)}| \leqslant \frac{\beta}{1-\beta} \sum_{i=1}^{n} |x_i^{(k)} - x_i^{(k-1)}|, \tag{5''}$$

respectively.

The iteration process is terminated when the indicated estimates show that the preassigned accuracy is achieved.

The initial vector $x^{(0)}$ may be chosen, generally speaking, arbitrarily. Sometimes we can put $x^{(0)} = f$. But it is most expedient to take approximate values of the unknowns (obtained by rough approximation) as components of the vector $x^{(0)}$.

System (1) can be reduced to the form (2) by various methods (see [12], [54]). It is only important that one of the conditions

(4) or (5) is fulfilled. Let us confine ourselves to considering two of such methods.

The first method. If the diagonal elements of matrix A are nonzero, i.e.

$$a_{ii} \neq 0 \quad (i = 1, 2, \ldots, n),$$

then system (1) can be written in the form

$$\left.\begin{aligned} x_1 &= \frac{1}{a_{11}}(b_1 - a_{12}x_2 - \ldots - a_{1n}x_n), \\ x_2 &= \frac{1}{a_{22}}(b_2 - a_{21}x_1 - a_{23}x_3 - \ldots - a_{2n}x_n), \\ &\cdots\cdots\cdots\cdots\cdots\cdots \\ x_n &= \frac{1}{a_{nn}}(b_n - a_{n1}x_1 - \ldots - a_{n,n-1}x_{n-1}). \end{aligned}\right\} \tag{6}$$

In this case the elements of matrix C are determined in the following way:

$$c_{ij} = -\frac{a_{ij}}{a_{ii}} \ (i \neq j), \quad c_{ii} = 0,$$

and then conditions (4) and (5) acquire the form

$$\sum_{\substack{j=1 \\ j \neq 1}}^{n} \left|\frac{a_{ij}}{a_{ii}}\right| \leqslant \alpha < 1 \quad (i = 1, 2, \ldots, n), \tag{7}$$

and

$$\sum_{\substack{i=1 \\ i \neq j}}^{n} \left|\frac{a_{ij}}{a_{ii}}\right| \leqslant \beta < 1 \quad (j = 1, 2, \ldots, n), \tag{8}$$

respectively. Inequalities (7), (8) will be fulfilled if the diagonal elements of matrix A meet the condition

$$|a_{ii}| > \sum_{j \neq i} |a_{ij}| \quad (i = 1, 2, \ldots, n), \tag{9}$$

i. e. if the moduli of the diagonal coefficients for each equation of the system exceed the sum of the moduli of all the remaining coefficients (without taking the constant terms into account.)

The second method is illustrated in Example 2 and requires no explanation.

Generally speaking, there exist convergent iteration methods of solving any system with a nonsingular matrix; but they are not always convenient for practical applications.

If an iteration method converges, then it provides the following advantages as compared with the methods considered above.

(1) If iterations converge sufficiently rapidly, i. e. if less than n iterations are enough to solve a system, then time is gained, since the number of arithmetical operations required for one iteration is proportional to n^2, and the total number of arithmetical operations in the Gaussian method, for instance, is proportional to n^3.

(2) Rounding off errors in the method of iteration are felt considerably less than in the Gaussian method. Besides, the method of iteration is *self-correcting*, i. e. an individual error made in the computations does not affect the final result, since an erroneous approximation may be considered as a new initial vector. The last circumstance is often used for specifying the values of the unknowns obtained by the Gaussian method (see Example 2 in Sec. 3.10).

(3) The method of iteration becomes especially advantageous in solving systems with a considerable number of coefficients equal to zero. Such systems appear, for instance, in solving equations in partial derivatives.

(4) The iteration process leads to performing uniform operations and is comparatively easily programmed on a computer.

Example 1. Solve the system, using the method of simple iteration:

$$\left.\begin{aligned} 20.9x_1 + 1.2x_2 - 2.1x_3 + 0.9x_4 &= 21.70,\\ 1.2x_1 + 21.2x_2 + 1.5x_3 + 2.5x_4 &= 27.46,\\ 2.1x_1 + 1.5x_2 + 19.8x_3 + 1.3x_4 &= 28.76,\\ 0.9x_1 + 2.5x_2 + 1.3x_3 + 32.1x_4 &= 49.72. \end{aligned}\right\} \qquad (10)$$

Solution. Reduce the system to the form (6):

$$x_1 = \frac{1}{20.9}(21.70 - 1.2x_2 - 2.1x_3 - 0.9x_4),$$

$$x_2 = \frac{1}{21.2}(27.46 - 1.2x_1 - 1.5x_3 - 2.5x_4),$$

$$x_3 = \frac{1}{19.8}(28.76 - 2.1x_1 - 1.5x_2 - 1.3x_4),$$

$$x_4 = \frac{1}{32.1}(49.72 - 0.9x_1 - 2.5x_2 - 1.3x_3).$$

Let us note here that the coefficients of the obtained system satisfy condition (4). Indeed,

$$\sum_{j=1}^{4} |c_{1j}| \approx 0.20 < 1, \quad \sum_{j=1}^{4} |c_{2j}| \approx 0.24 < 1,$$

$$\sum_{j=1}^{4} |c_{3j}| \approx 0.25 < 1, \quad \sum_{j=1}^{4} |c_{4j}| \approx 0.15 < 1.$$

Thus, convergence of iterations is guaranteed. Since $a = 0.25$, the accuracy of the kth approximation can be estimated by formula (4″) for $\frac{\alpha}{1-\alpha} = \frac{1}{3}$.

As the initial vector $x^{(0)}$ let us take the elements of the column of constant terms rounding their values to two decimal digits:

$$x^{(0)} = \begin{pmatrix} 1.04 \\ 1.30 \\ 1.45 \\ 1.55 \end{pmatrix}.$$

We shall keep computing until the quantities $|x_i^{(k)} - x_i^{(k-1)}|$ $(i = 1, 2, 3, 4)$ become less than $\varepsilon = 10^{-3}$.

We compute in succession:

for $k = 1$

$$x_1^{(1)} = \frac{1}{20.9}(21.70 - 1.560 - 3.045 - 1.395) = 0.75,$$

$$x_2^{(1)} = \frac{1}{21.2}(27.46 - 1.248 - 2.175 - 3.875) = 0.95,$$

$$x_3^{(1)} = \frac{1}{19.8}(28.76 - 2.184 - 1.950 - 2.015) = 1.14,$$

$$x_4^{(1)} = \frac{1}{32.1}(49.72 - 0.936 - 3.250 - 1.885) = 1.36;$$

for $k = 2$

$$x_1^{(2)} = \frac{16.942}{20.9} = 0.8106, \quad x_3^{(2)} = \frac{23.992}{19.8} = 1.2117,$$

$$x_2^{(2)} = \frac{21.450}{21.2} = 1.0118, \quad x_4^{(2)} = \frac{45.188}{32.1} = 1.4077;$$

for $k = 3$

$$x_1^{(3)} = \frac{16.67434}{20.9} = 0.7978, \quad x_3^{(3)} = \frac{23.71003}{19.8} = 1.1975,$$

$$x_2^{(3)} = \frac{21.15048}{21.2} = 0.9977, \quad x_4^{(3)} = \frac{44.88575}{32.1} = 1.3983;$$

for $k = 4$

$$x_1^{(4)} = \frac{16.7295}{20.9} = 0.8004, \quad x_3^{(4)} = \frac{23.7703}{19.8} = 1.2005,$$

$$x_2^{(4)} = \frac{21.2106}{21.2} = 1.0005, \quad x_4^{(4)} = \frac{44.9510}{32.1} = 1.4003.$$

Calculate the moduli of the differences of the values of $x_i^{(k)}$ for

$k=3$ and $k=4$:

$$|x_1^{(3)}-x_1^{(4)}|=0.0026, \quad |x_3^{(3)}-x_3^{(4)}|=0.0030,$$
$$|x_2^{(3)}-x_2^{(4)}|=0.0028, \quad |x_4^{(3)}-x_4^{(4)}|=0.0020.$$

Since all of them exceed the preassigned number $\varepsilon=10^{-3}$, the process of iteration is continued.

We get for $k=5$:

$$x_1^{(5)}=\frac{16.71808}{20.9}=0.7999, \quad x_3^{(5)}=\frac{23.75802}{19.8}=1.1999,$$
$$x_2^{(5)}=\frac{21.19802}{21.2}=0.9999, \quad x_4^{(5)}=\frac{44.93774}{32.1}=1.3999.$$

Find the moduli of the differences of the values of $x_i^{(k)}$ for $k=4$ and $k=5$:

$$|x_1^{(4)}-x_1^{(5)}|=0.0005, \quad |x_3^{(4)}-x_3^{(5)}|=0.0006,$$
$$|x_2^{(4)}-x_2^{(5)}|=0.0006, \quad |x_4^{(4)}-x_4^{(5)}|=0.0004.$$

They are less than the given number ε, therefore we take the following as the solution

$$x_1\approx 0.7999, \quad x_2\approx 0.9999, \quad x_3\approx 1.1999, \quad x_4\approx 1.3999.$$

In accordance with estimate (4″) the errors of these values should not exceed $\frac{1}{3}\cdot 0.0006=0.0002$.

For comparison we give the exact values of the unknowns:

$$x_1=0.8, \quad x_2=1.0, \quad x_3=1.2, \quad x_4=1.4.$$

Example 2. Solve the system

$$\left.\begin{aligned} 1.02x_1-0.05x_2-0.10x_3&=0.795,\\ -0.11x_1+1.03x_2-0.05x_3&=0.849,\\ -0.11x_1-0.12x_2+1.04x_3&=1.398, \end{aligned}\right\} \tag{11}$$

carrying out three iterations. Indicate the error of the result obtained.

Solution. The matrix of this system is such that the diagonal elements are close to unity, and all the remaining ones are considerably less than unity. Therefore, for the method of iteration to be used it is natural to rewrite system (11) in the form

$$\left.\begin{aligned} x_1&=0.795-0.02x_1+0.05x_2+0.10x_3,\\ x_2&=0.849+0.11x_1-0.03x_2+0.05x_3,\\ x_3&=1.398+0.11x_1+0.12x_2-0.04x_3. \end{aligned}\right\}$$

The conditions of convergence (4) for the obtained system are

fulfilled. Indeed,

$$\sum_{j=1}^{3} |c_{1j}| = 0.02 + 0.05 + 0.10 = 0.17 < 1,$$

$$\sum_{j=1}^{3} |c_{2j}| = 0.11 + 0.03 + 0.05 = 0.19 < 1,$$

$$\sum_{j=1}^{3} |c_{3j}| = 0.11 + 0.12 + 0.04 = 0.27 < 1.$$

As the initial vector $x^{(0)}$ we take the column of constant terms rounding its elements to two decimal places:

$$x^{(0)} = \begin{pmatrix} 0.80 \\ 0.85 \\ 1.40 \end{pmatrix}.$$

Then we successively find:
for $k = 1$

$$x_1^{(1)} = 0.795 - 0.016 + 0.0425 + 0.140 = 0.9615 \approx 0.962,$$
$$x_2^{(1)} = 0.849 + 0.088 - 0.0255 + 0.070 = 0.9815 \approx 0.982,$$
$$x_3^{(1)} = 1.398 + 0.088 + 0.1020 - 0.056 = 1.532;$$

for $k = 2$

$$x_1^{(2)} = 0.97806 \approx 0.978, \quad x_2^{(2)} = 1.00196 \approx 1.002,$$
$$x_3^{(2)} = 1.56038 \approx 1.560;$$

for $k = 3$

$$x_1^{(3)} = 0.980, \quad x_2^{(3)} = 1.004, \quad x_3^{(3)} = 1.563.$$

The values of the unknowns for $k = 2$ and $k = 3$ differ by not more than $3 \cdot 10^{-3}$; therefore, if

$$x_1 \approx 0.980, \quad x_2 \approx 1.004, \quad x_3 \approx 1.563$$

are taken as approximate values of the unknowns, then their error will not exceed

$$\frac{0.27}{1 - 0.27} \cdot 3 \cdot 10^{-3} < 1.1 \cdot 10^{-3}.$$

Problems

Solve the following systems using the method of simple iteration. Continue the iteration process until the difference between two successive approximations of the unknowns $x_i^{(k)}$ becomes less than the indicated number ε.

In Problems 1 and 2 compare the answer with the given exact values of the unknowns.

1. $A=\begin{pmatrix} 1.02 & -0.25 & -0.30 \\ -0.41 & 1.13 & -0.15 \\ -0.25 & -0.14 & 1.21 \end{pmatrix}$, $\boldsymbol{b}=\begin{pmatrix} 0.515 \\ 1.555 \\ 2.780 \end{pmatrix}$, $\boldsymbol{x}=\begin{pmatrix} 2.0 \\ 2.5 \\ 3.0 \end{pmatrix}$,

$\varepsilon=10^{-3}$.

2. $A=\begin{pmatrix} 10.9 & 1.2 & 2.1 & 0.9 \\ 1.2 & 11.2 & 1.5 & 2.5 \\ 2.1 & 1.5 & 9.8 & 1.3 \\ 0.9 & 2.5 & 1.3 & 12.1 \end{pmatrix}$, $\boldsymbol{b}=\begin{pmatrix} -7.0 \\ 5.3 \\ 10.3 \\ 24.6 \end{pmatrix}$, $\boldsymbol{x}=\begin{pmatrix} -1 \\ 0 \\ 1 \\ 2 \end{pmatrix}$,

$\varepsilon=10^{-3}$.

3. $A=\begin{pmatrix} 4.00 & 0.24 & -0.08 & 0.16 \\ 0.09 & 3.00 & -0.15 & -0.12 \\ 0.04 & -0.08 & 4.00 & 0.06 \\ 0.02 & 0.06 & 0.04 & -10.00 \end{pmatrix}$, $\boldsymbol{b}=\begin{pmatrix} 8 \\ 9 \\ 20 \\ 1 \end{pmatrix}$,

$\varepsilon=0.5\cdot 10^{-3}$.

4. $A=\begin{pmatrix} 8.714 & 2.180 & 5.684 \\ -1.351 & 10.724 & 5.224 \\ 2.489 & -0.459 & 6.799 \end{pmatrix}$, $\boldsymbol{b}=\begin{pmatrix} 49.91 \\ 50.17 \\ 32.68 \end{pmatrix}$, $\varepsilon=10^{-4}$.

5. $A=\begin{pmatrix} 24.21+\alpha & 2.42 & 3.85 \\ 2.31 & 31.49 & 1.52 \\ 3.49 & 4.85 & 28.72+\alpha \end{pmatrix}$, $\boldsymbol{b}=\begin{pmatrix} 30.24 \\ 40.95-\beta \\ 42.81 \end{pmatrix}$,

$\varepsilon=10^{-4}$, $\alpha=0.2\cdot k$, $k=0, 1, \ldots, 4$, $\beta=0.2\cdot k$, $k=0, 1, \ldots, 5$.

3.10 THE SEIDEL METHOD

The Seidel method is a certain modification of the method of simple iteration. The principal idea behind it is that in computing the $(k+1)$th approximation of the unknown x_i for $i>1$, the earlier computed $(k+1)$th approximations of the unknowns x_1, x_2, ..., x_{i-1} are taken into account. Thus for system (2) of Sec. 3.9 computations by the Seidel method are carried out using the following formulas

$$\left.\begin{aligned} x_1^{(k+1)} &= c_{11}x_1^{(k)}+c_{12}x_2^{(k)}+\ldots+c_{1n}x_n^{(k)}+f_1, \\ x_2^{(k+1)} &= c_{21}x_1^{(k+1)}+c_{22}x_2^{(k)}+\ldots+c_{2n}x_n^{(k)}+f_2, \\ &\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ x_n^{(k+1)} &= c_{n1}x_1^{(k+1)}+c_{n2}x_2^{(k+1)}+\ldots+c_{n,n-1}x_{n-1}^{(k+1)}+c_{nn}x_n^{(k)}+f_n. \end{aligned}\right\} \quad (1)$$

Note that the convergence conditions given above (see Sec. 3.9) for simple iteration remain valid for iteration by the Seidel me-

thod. Ordinarily, the Seidel method yields a better convergence than does the method of simple iteration, though it is not always the case (see [54]). Besides, the Seidel method turns out to be more convenient for programming, since in computing $x_i^{(k+1)}$ there is no need to store the values $x_1^{(k)}, \ldots, x_{i-1}^{(k)}$.

Recommendations for applying the Seidel method remain the same as for the method of simple iteration.

Example 1. Solve the system (10) of Sec. 3.9 by the Seidel method.

Solution. In Example 1 (Sec. 3.9) this system was already reduced to the form (6) (Sec. 3.9), the initial vector $\boldsymbol{x}^{(0)}$ was also chosen there. Applying the Seidel process, we successively obtain:

For $k=1$

$$x_1^{(1)} = \frac{1}{20.9}(21.70 - 1.560 - 3.045 - 1.395) = 0.7512.$$

In computing $x_2^{(1)}$ we use the obtained value $x_1^{(1)}$:

$$x_2^{(1)} = \frac{1}{21.2}(27.46 - 0.900 - 2.175 - 3.875) = 0.9674.$$

In computing $x_3^{(1)}$ we use the values $x_1^{(1)}$ and $x_2^{(1)}$:

$$x_3^{(1)} = \frac{1}{19.8}(28.76 - 1.575 - 1.455 - 2.015) = 1.1977.$$

Finally, using the values $x_1^{(1)}$, $x_2^{(1)}$, $x_3^{(1)}$, we get

$$x_4^{(1)} = \frac{1}{32.1}(49.72 - 0.675 - 2.425 - 1.560) = 1.4037.$$

For $k=2$ and $k=3$ computations are carried out in a similar way.

We get;

for $k=2$

$$x_1^{(2)} = \frac{16.76062}{20.9} = 0.8019, \quad x_3^{(2)} = \frac{23.75180}{19.8} = 1.9996,$$

$$x_2^{(2)} = \frac{21.19202}{21.2} = 0.9996, \quad x_4^{(2)} = \frac{44.93981}{32.1} = 1.4000;$$

for $k=3$

$$x_1^{(3)} = \frac{16.72132}{20.9} = 0.80006, \quad x_3^{(3)} = \frac{23.759844}{19.8} = 1.19999,$$

$$x_2^{(3)} = \frac{21.200528}{21.2} = 1.00002, \quad x_4^{(3)} = \frac{44.939909}{32.1} = 1.40000.$$

Table 16 contains the values of the unknowns of system (10) of Sec. 3.9 obtained as a result of the third iteration both by the Seidel method and the method of simple iteration; exact values are also listed. The comparison shows that here the Seidel method leads to the goal more rapidly.

Table 16

Values of Unknowns of System (10) (from Sec. 3.9)

	x_1	x_2	x_3	x_4
Method of simple iteration	0.7978	0.9977	1.1975	1.3983
Seidel method	0.80006	1.00002	1.19999	1.40000
Exact values	0.8	1.0	1.2	1.4

Example 2. For the system

$$\left.\begin{aligned} 6x_1 - x_2 - x_3 &= 11.33, \\ -x_1 + 6x_2 - x_3 &= 32, \\ -x_1 - x_2 + 6x_3 &= 42, \end{aligned}\right\} \qquad (2)$$

the approximate values of the unknowns obtained with three significant digits by the Gaussian method are known:

$$x_1 \approx 4.67, \quad x_2 \approx 7.62, \quad x_3 \approx 9.05.$$

Applying the Seidel method, correct the solutions so that the values of the unknowns $x_i^{(k)}$ and $x_i^{(k+1)}$ differed by not more than $5 \cdot 10^{-4}$.

Solution. Reduce system (2) to the form

$$x_1 = \frac{1}{6}(11.33 + x_2 + x_3),$$

$$x_2 = \frac{1}{6}(32 + x_1 + x_3),$$

$$x_3 = \frac{1}{6}(42 + x_1 + x_2).$$

Taking as the initial approximation the values obtained by the Gaussian method

$$x_1^{(0)} = 4.67, \; x_2^{(0)} = 7.62, \; x_3^{(0)} = 9.05,$$

we get:

for $k = 1$

$$x_1^{(1)} = \frac{1}{6}(11.33 + 16.67) = 4.66667,$$

$$x_2^{(1)} = \frac{1}{6}(32 + 13.71667) = 7.61944,$$

$$x_3^{(1)} = \frac{1}{6}(42 + 12.28611) = 9.04768;$$

for $k=2$

$$x_1^{(2)} = \frac{1}{6}(11.33 + 16.66712) = 4.66619,$$

$$x_2^{(2)} = \frac{1}{6}(32 + 13.71387) = 7.61897,$$

$$x_3^{(2)} = \frac{1}{6}(42 + 12.28516) = 9.04752.$$

Since for the above system the condition (4) of Sec. 3.9 is fulfilled for $\alpha = \frac{1}{3}$, the obtained approximation has the error not exceeding $\frac{1}{2} \cdot 5 \cdot 10^{-4} = 2.5 \cdot 10^{-4}$.

Thus, the following may be taken as the solutions:

$$x_1 \approx 4.666, \quad x_2 \approx 7.619, \quad x_3 \approx 9.048.$$

Problems

Solve systems 1 and 2 both by the method of simple iteration and by the Seidel method. Compare the rapidities of convergence of the iterations. Compare the obtained values with the indicated exact values of the unknowns.

1. $A = \begin{pmatrix} 6.1 & 2.2 & 1.2 \\ 2.2 & 5.5 & -1.5 \\ 1.2 & -1.5 & 7.2 \end{pmatrix}$, $b = \begin{pmatrix} 16.55 \\ 10.55 \\ 16.80 \end{pmatrix}$, $x = \begin{pmatrix} 1.5 \\ 2.0 \\ 2.5 \end{pmatrix}$.

2. $A = \begin{pmatrix} 3.82 & 1.02 & 0.75 & 0.81 \\ 1.05 & 4.53 & 0.98 & 1.53 \\ 0.73 & 0.85 & 4.71 & 0.81 \\ 0.88 & 0.81 & 1.28 & 3.50 \end{pmatrix}$, $b = \begin{pmatrix} 15.655 \\ 22.705 \\ 23.480 \\ 16.110 \end{pmatrix}$, $x = \begin{pmatrix} 2.5 \\ 3.0 \\ 3.5 \\ 2.0 \end{pmatrix}$.

3. Solve the systems from Problems 3 to 5 (Sec. 3.9) by the Seidel method.

3.11 APPLYING THE METHOD OF ITERATION FOR CORRECTING ELEMENTS OF AN INVERSE MATRIX

Let approximate values of the elements of an inverse matrix A^{-1} be obtained for a nonsingular matrix A. Denote the matrix with such elements by $D_0 \approx A^{-1}$; for improving the elements of the inverse matrix form the following iteration process:

$$F_{k-1} = E - AD_{k-1} \quad (k = 1, 2, \ldots), \tag{1}$$

$$D_k = D_{k-1}(E + F_{k-1}) \quad (k = 1, 2, \ldots). \tag{2}$$

It is proved (see [2], [12]) that the iterations converge if the initial matrix D_0 is sufficiently close to the initial matrix A^{-1}.

Iterations are usually continued until the moduli of the elements of matrix F_k are less than the specified number ε, and then we approximately put

$$A^{-1} \approx D_k.$$

The indicated iteration process turns out quite useful, since the exact methods of computing an inverse matrix often give notable errors due to inevitable rounding and a large number of arithmetical operations involved.

Example 1. Correct the elements of the approximate inverse matrix obtained in Example 1 of Sec. 3.8. Proceed with the iterations until the moduli of the elements of matrix F_k are less than, or equal to, $5 \cdot 10^{-5}$.

Solution. Using the Gaussian method we obtained the approximate inverse for the matrix (3) (see Sec. 3.8). Let us take it for the initial approximation D_0:

$$D_0 = \begin{pmatrix} -0.21121 & -0.46003 & 0.16284 & 0.26956 \\ -0.03533 & 0.16873 & 0.01573 & -0.08920 \\ 0.23030 & 0.04607 & -0.00944 & -0.19885 \\ -0.29316 & -0.38837 & 0.06128 & 0.18513 \end{pmatrix}.$$

Find the matrix

$$F_0 = E - AD_0.$$

The product AD_0 was computed on page 80. Subtracting it from the unit matrix, we get

$$F_0 = E - AD_0 = 10^{-3} \cdot \begin{pmatrix} 0.03 & 0.00 & 0.01 & 0.00 \\ 0.25 & 0.03 & 0.02 & 0.39 \\ 8.08 & 10.17 & 0.18 & -0.09 \\ 0.00 & 0.00 & 0.00 & -0.48 \end{pmatrix}.$$

Now we find the product D_0F_0:

$$D_0F_0 = \begin{pmatrix} -0.21121 & -0.46003 & 0.16284 & 0.26956 \\ -0.03533 & 0.16873 & 0.01573 & -0.08920 \\ 0.23030 & 0.04607 & -0.00944 & -0.19885 \\ -0.29316 & -0.38837 & 0.06128 & 0.18513 \end{pmatrix} \times$$

$$\times 10^{-3} \cdot \begin{pmatrix} 0.03 & 0.00 & 0.01 & 0.00 \\ 0.25 & 0.03 & 0.02 & 0.39 \\ 8.08 & 10.17 & 0.18 & -0.09 \\ 0.00 & 0.00 & 0.00 & -0.48 \end{pmatrix} =$$

$$= 10^{-3} \cdot \begin{pmatrix} 1.19 & 1.64 & 0.02 & -0.32 \\ 0.17 & 0.16 & 0.01 & 0.11 \\ -0.06 & -0.09 & 0.00 & 0.11 \\ 0.39 & 0.61 & 0.00 & -0.24 \end{pmatrix}.$$

Then, by formula (2) we find for $k=1$

$$D_1 = D_0 + D_0F_0 = \begin{pmatrix} -0.21121 & -0.46003 & 0.16284 & 0.26956 \\ -0.03533 & 0.16873 & 0.01573 & -0.08920 \\ 0.23030 & 0.04607 & -0.00944 & -0.19885 \\ -0.29316 & -0.38837 & 0.06128 & 0.18513 \end{pmatrix} +$$

$$+ 10^{-3} \cdot \begin{pmatrix} 1.19 & 1.64 & 0.02 & -0.32 \\ 0.17 & 0.16 & 0.01 & 0.11 \\ -0.06 & -0.09 & 0.00 & 0.11 \\ 0.39 & 0.61 & 0.00 & -0.24 \end{pmatrix} =$$

$$= \begin{pmatrix} -0.21002 & -0.45839 & 0.16286 & 0.26924 \\ -0.03516 & 0.16889 & 0.01574 & -0.08909 \\ 0.23024 & 0.04598 & -0.00944 & -0.19874 \\ -0.29277 & -0.38776 & 0.06128 & 0.18489 \end{pmatrix}.$$

To check whether the preassigned accuracy is reached compute the product of the matrices AD_1:

$$AD_1 = E - 10^{-5} \cdot \begin{pmatrix} 2 & -2 & 1 & 3 \\ 0 & 2 & -1 & 0 \\ 3 & 4 & -5 & 1 \\ 1 & 0 & 0 & 1 \end{pmatrix}.$$

Then

$$F_1 = E - AD_1 = 10^{-5} \cdot \begin{pmatrix} 2 & -2 & 1 & 3 \\ 0 & 2 & -1 & 0 \\ 3 & 4 & -5 & 1 \\ 1 & 0 & 0 & 1 \end{pmatrix}.$$

Since the numerically greatest element of the matrix F_1 is equal to $5 \cdot 10^{-5}$, we may write with the specified accuracy

$$A^{-1} \approx D_1.$$

Problems

1. For the matrix

$$A = \begin{pmatrix} 24.21+\alpha & 2.42 & 3.85 \\ 2.31 & 31.49 & 1.52 \\ 3.49 & 4.85 & 28.72+\alpha \end{pmatrix}, \quad \alpha = 0.2 \cdot n, \; n = 0, 1, \ldots, 10,$$

find the inverse A^{-1}. Take the inverse matrix obtained in Problem 5 of Sec. 3.8 as the initial approximation, rounding off its elements to three decimal places. Continue the iteration process until the moduli of the elements of the matrix F_k become less than 10^{-5}.

4

NUMERICAL SOLUTION OF SYSTEMS OF NONLINEAR EQUATIONS

4.1 NEWTON'S METHOD FOR A SYSTEM OF TWO EQUATIONS

Suppose we are given a system

$$\left.\begin{aligned} F(x,\ y) &= 0, \\ G(x,\ y) &= 0. \end{aligned}\right\} \tag{1}$$

According to Newton's method successive approximations are computed by the following formulas:

$$\left.\begin{aligned} x_{n+1} &= x_n - \frac{1}{J(x_n,\ y_n)}\begin{vmatrix} F(x_n,\ y_n) & F'_y(x_n,\ y_n) \\ G(x_n,\ y_n) & G'_y(x_n,\ y_n) \end{vmatrix} = x_n - \frac{\Delta_x^{(n)}}{J(x_n,\ y_n)}, \\ y_{n+1} &= y_n - \frac{1}{J(x_n,\ y_n)}\begin{vmatrix} F'_x(x_n,\ y_n) & F(x_n,\ y_n) \\ G'_x(x_n,\ y_n) & G(x_n,\ y_n) \end{vmatrix} = y_n - \frac{\Delta_y^{(n)}}{J(x_n,\ y_n)}, \end{aligned}\right\} \tag{2}$$

where

$$\Delta_x^{(n)} = \begin{vmatrix} F(x_n,\ y_n) & F'_y(x_n,\ y_n) \\ G(x_n,\ y_n) & G'_y(x_n,\ y_n) \end{vmatrix}, \quad \Delta_y^{(n)} = \begin{vmatrix} F'_x(x_n,\ y_n) & F(x_n,\ y_n) \\ G'_x(x_n,\ y_n) & G(x_n,\ y_n) \end{vmatrix},$$

and the Jacobian

$$J(x,\ y) = \begin{vmatrix} F'_x(x,\ y) & F'_y(x,\ y) \\ G'_x(x,\ y) & G'_y(x,\ y) \end{vmatrix} \neq 0.$$

The initial approximations x_0, y_0 are determined roughly.

Newton's method is effective only when the initial approximation is sufficiently close to the solution of a system.

Example 1. Find the real roots of the system

$$F(x,\ y) \equiv 2x^3 - y^2 - 1 = 0,$$
$$G(x,\ y) \equiv xy^3 - y - 4 = 0.$$

Solution. Find graphically the approximate values $x_0 = 1.2$ and

$y_0 = 1.7$. Computing the Jacobian at the point (1.2; 1.7), we have

$$J(x, y) = \begin{vmatrix} 6x^2 - 2y & \\ y^3 & 3xy^2 - 1 \end{vmatrix},$$

$$J(1.2;\ 1.7) = \begin{vmatrix} 8.64 & -3.40 \\ 4.91 & 9.40 \end{vmatrix} = 97.910.$$

From formulas (2) we get

$$x_1 = 1.2 - \frac{1}{97.910}\begin{vmatrix} -0.434 & -3.40 \\ 0.1956 & 9.40 \end{vmatrix} = 1.2 + 0.0349 = 1.2349,$$

$$y_1 = 1.7 - \frac{1}{97.910}\begin{vmatrix} 8.64 & -0.434 \\ 4.91 & 0.1956 \end{vmatrix} = 1.7 - 0.0390 = 1.6610.$$

Continuing this process with the obtained values x_1 and y_1 we get

$$x_2 = 1.2343, \quad y_2 = 1.6615 \text{ and so forth.}$$

Example 2. Solve the following system of equations

$$F(x, y) \equiv \cos(0.4y + x^2) + x^2 + y^2 - 1.6 = 0,$$

$$G(x, y) \equiv 1.5x^2 - \frac{y^2}{0.36} - 1 = 0.$$

Solution. We find approximate values graphically

$$x_0 = 1.04, \quad y_0 = 0.47.$$

Further approximations are found from formulas (2):

$$x_{n+1} = x_n - \frac{\Delta_x^{(n)}}{J(x_n, y_n)},$$

$$y_{n+1} = y_n - \frac{\Delta_y^{(n)}}{J(x_n, y_n)}.$$

The results of the computations are tabulated below.

Answer. $x = 1.03864$, $y = 0{,}47173$.

Table 17

x	1.04	1.03864	F'_y	0.55801	0.56172	J	−1.98549	−1.99889
y	0.47	0.47173	G'_x	3.12	3.11592	$-\frac{\Delta_x}{J}$	−0.00136	0.00000
F	−0.00084	0.00000	G'_y	−2.61111	−2.62072			
G	0.00879	0.00002	Δx	−0.00271	0.00001	$-\frac{\Delta_y}{J}$	0.00173	0.00000
F'_x	0.09364	0.09483	Δy	+0.00344	0.00000			

Problems

Solve the following systems by Newton's method. Obtain the results with five correct digits. Find the initial approximations graphically.

1. $\left.\begin{aligned}\tan(xy+k) &= x^2,\\ \alpha x^2+2y^2 &= 1,\end{aligned}\right\}$ $x>0,\ y>0,\ \alpha=0.5+0.1\cdot m,$
$k=0.1\cdot m\ (m=0,\ 1,\ \ldots,\ 4).$

2. $\left.\begin{aligned}e^{xy} &= x^2-y+\alpha,\\ (x+0.5)^2+y^2 &= k,\end{aligned}\right\}$ $x>0,\ y>0,\ \alpha=1+0.1\cdot m,$
$k=0.6+0.1\cdot m\ (m=0,\ 1,\ \ldots,\ 4).$

4.2 THE METHOD OF SIMPLE ITERATION FOR A SYSTEM OF TWO EQUATIONS

Let there be given a system of two equations in two unknowns:

$$\left.\begin{aligned}F_1(x,\ y)&=0,\\ F_2(x,\ y)&=0,\end{aligned}\right\} \tag{1}$$

whose real roots are required to be found with a specified degree of accuracy.

Suppose, system (1) permits only isolated roots. Their number and approximate values can be found by constructing the curves $F_1(x,\ y)=0$ and $F_2(x,\ y)=0$ and determining the coordinates of the points at which they intersect.

For applying the method of iteration the system (1) is reduced to the form

$$\left.\begin{aligned}x&=\varphi_1(x,\ y),\\ y&=\varphi_2(x,\ y).\end{aligned}\right\} \tag{2}$$

The functions $\varphi_1(x,\ y)$ and $\varphi_2(x,\ y)$ are called *iterative*. The algorithm of solution is given by the formulas

$$\left.\begin{aligned}x_{n+1}&=\varphi_1(x_n,\ y_n),\\ y_{n+1}&=\varphi_2(x_n,\ y_n)\quad (n=0,\ 1,\ 2,\ \ldots),\end{aligned}\right\} \tag{3}$$

where $x_0,\ y_0$ is some initial approximation.

There exists the following theorem: *Let in some closed neighbourhood* $R\ (a\leqslant x\leqslant A,\ b\leqslant y\leqslant B)$ *there be one and only one solution* $x=\xi,\ y=\eta$ *for the system* (2). *If*

(1) *the functions* $\varphi_1(x,\ y)$ *and* $\varphi_2(x,\ y)$ *are defined and continuously differentiable in* R,

(2) *the initial approximations* $x_0,\ y_0$ *and all successive approximations* $x_n,\ y_n\ (n=1,\ 2,\ \ldots)$ *belong to* R,

(3) *the following inequalities are fulfilled in* R

$$\left.\begin{aligned}\left|\frac{\partial\varphi_1}{\partial x}\right|+\left|\frac{\partial\varphi_2}{\partial x}\right|\leqslant q_1<1,\\ \left|\frac{\partial\varphi_1}{\partial y}\right|+\left|\frac{\partial\varphi_2}{\partial y}\right|\leqslant q_2<1,\end{aligned}\right\} \tag{4}$$

then the process of successive approximations (3) *converges to the solution* $x=\xi$, $y=\eta$ *of the system, i. e.*

$$\lim_{n\to\infty} x_n=\xi \text{ and } \lim_{n\to\infty} y_n=\eta.$$

This theorem remains true if condition (4) is replaced by the condition

$$\left.\begin{aligned}\left|\frac{\partial\varphi_1}{\partial x}\right|+\left|\frac{\partial\varphi_1}{\partial y}\right|\leqslant q_1<1,\\ \left|\frac{\partial\varphi_2}{\partial x}\right|+\left|\frac{\partial\varphi_2}{\partial y}\right|\leqslant q_2<1.\end{aligned}\right\} \tag{4'}$$

Error estimate of the nth approximation is given by the inequality

$$|\xi-x_n|+|\eta-y_n|\leqslant\frac{M}{1-M}(|x_n-x_{n-1}|+|y_n-y_{n-1}|),$$

where M is the greatest number of the numbers q_1, q_2 entering the inequalities (4) or (4'). Convergence of the method of iteration is considered as good if $M<\frac{1}{2}$, and $\frac{M}{1-M}<1$, so that if the first three decimal digits coincide in two successive approximations, then the error of the last approximation does not exceed 0.001.

Example 1. For the system

$$x^3+y^3-6x+3=0,\quad x^3-y^3-6y+2=0$$

find the positive roots with three correct digits.

Solution. For applying the method of iteration rewrite the given system as:

$$x=\frac{x^3+y^3}{6}+\frac{1}{2}\equiv\varphi_1(x,\ y),\quad y=\frac{x^3-y^3}{6}+\frac{1}{3}\equiv\varphi_2(x,\ y).$$

Consider the square $0\leqslant x\leqslant 1$, $0\leqslant y\leqslant 1$. If point $(x_0,\ y_0)$ is situated in this square, then we have

$$0<\varphi_1(x_0,\ y_0)<1 \text{ and } 0<\varphi_2(x_0,\ y_0)<1.$$

Since $0<\frac{(x_0^3+y_0^3)}{6}<\frac{1}{3}$, $-\frac{1}{6}<\frac{(x_0^3-y_0^3)}{6}<\frac{1}{6}$, the sequence $(x_k,\ y_k)$ remains in a square for any choice of the point $(x_0,\ y_0)$. Moreover, the points $(x_k,\ y_k)$ remain in the rectangle $\frac{1}{2}<x<\frac{5}{6}$,

$\frac{1}{6} < y < \frac{1}{2}$ $\left(\text{since } \frac{1}{3}+\frac{1}{2}=\frac{5}{6},\ \frac{1}{3}-\frac{1}{6}=\frac{1}{6},\ \frac{1}{3}+\frac{1}{6}=\frac{1}{2}\right)$. For the points of this rectangle we have

$$\left|\frac{\partial\varphi_1}{\partial x}\right| + \left|\frac{\partial\varphi_1}{\partial y}\right| = \frac{x^2}{2} + \frac{y^2}{2} < \frac{25/36 + 1/4}{2} = \frac{34}{72} < 1,$$

$$\left|\frac{\partial\varphi_2}{\partial x}\right| + \left|\frac{\partial\varphi_2}{\partial y}\right| = \frac{x^2}{2} + \left|-\frac{y^2}{2}\right| < \frac{34}{72} < 1.$$

Hence, there exists the unique solution in the indicated rectangle and it can be found by the method of iteration. Putting $x_0 = \frac{1}{2}$, $y_0 = \frac{1}{2}$, we have

$$x_1 = \frac{1}{2} + \frac{1/8 + 1/8}{6} = 0.542, \quad x_2 = \frac{1}{2} + \frac{0.19615}{6} = 0.533,$$
$$y_1 = \frac{1}{3} + \frac{1/8 - 1/8}{6} = 0.333, \quad y_2 = \frac{1}{3} + \frac{0.1223}{6} = 0.354.$$

Continuing this process, we get

$$x_3 = 0.533, \quad x_4 = 0.532, \quad y_3 = 0.351, \quad y_4 = 0.351.$$

Since here $q_1 = q_2 = \frac{34}{72} < 0.5$, the coincidence of the first three decimal digits means that the required accuracy is achieved.

Thus, we can take $\xi = 0.532$, $\eta = 0.351$.

Note. Instead of the considered iteration process (3) sometimes it is more convenient to use the so-called Seidel process:

$$x_{n+1} = \varphi_1(x_n, y_n), \quad y_{n+1} = \varphi_2(x_{n+1}, y_n) \quad (n = 0, 1, 2, \ldots).$$

Constructing iterative functions for the system (1). We can recommend the following method for transforming the system of equations (1) to the form (2) with the condition (4) fulfilled. Put

$$\varphi_1(x, y) = x + \alpha F_1(x, y) + \beta F_2(x, y),$$
$$\varphi_2(x, y) = y + \gamma F_1(x, y) + \delta F_2(x, y) \quad (\alpha\delta \neq \beta\gamma).$$

Find the coefficients α, β, γ, δ as approximate solutions of the following system of equations:

$$\left.\begin{aligned} 1 + \alpha\frac{\partial F_1(x_0, y_0)}{\partial x} + \beta\frac{\partial F_2(x_0, y_0)}{\partial x} &= 0, \\ \alpha\frac{\partial F_1(x_0, y_0)}{\partial y} + \beta\frac{\partial F_2(x_0, y_0)}{\partial y} &= 0, \\ \gamma\frac{\partial F_1(x_0, y_0)}{\partial x} + \delta\frac{\partial F_2(x_0, y_0)}{\partial x} &= 0, \\ 1 + \gamma\frac{\partial F_1(x_0, y_0)}{\partial y} + \delta\frac{\partial F_2(x_0, y_0)}{\partial y} &= 0. \end{aligned}\right\} \tag{5}$$

With such a choice of the parameters the condition (4) will be met if the partial derivatives of the functions $F_1(x, y)$ and $F_2(x, y)$ vary not very rapidly in the neighbourhood (x_0, y_0).

Example 2. Choose suitable iterative functions $\varphi_1(x, y)$ and $\varphi_2(x, y)$ for the system of equations

$$x^2+y^2-1=0, \quad x^3-y=0$$

for $x_0=0.8$, $y_0=0.55$.

Solution. We shall look for the functions φ_1 and φ_2 in the form

$$\varphi_1(x, y)=x+\alpha(x^2+y^2-1)+\beta(x^3-y),$$
$$\varphi_2(x, y)=y+\gamma(x^2+y^2-1)+\delta(x^3-y).$$

To determine the parameters α, β, γ, δ form system (5). We have

$$\frac{\partial F_1}{\partial x}=2x, \quad \frac{\partial F_1(x_0, y_0)}{\partial x}=1.6, \quad \frac{\partial F_1}{\partial y}=2y, \quad \frac{\partial F_1(x_0, y_0)}{\partial y}=1.1,$$
$$\frac{\partial F_2}{\partial x}=3x^2, \quad \frac{\partial F_2(x_0, y_0)}{\partial x}=1.92, \quad \frac{\partial F_2}{\partial y}=-1, \quad \frac{\partial F_2(x_0, y_0)}{\partial y}=-1.$$

Hence we get the system

$$\begin{aligned} 1+1.6\alpha+1.92\beta&=0,\\ 1.1\alpha-\beta&=0,\\ 1.6\gamma+1.92\delta&=0,\\ 1+1.1\gamma-\delta&=0. \end{aligned}$$

Solving it, we obtain

$$\alpha\approx-0.3, \quad \gamma\approx-0.5, \quad \beta\approx-0.3, \quad \delta\approx0.4.$$

Thus,

$$\varphi_1(x, y)=x-0.3(x^2+y^2-1)-0.3(x^3-y),$$
$$\varphi_2(x, y)=y-0.5(x^2+y^2-1)+0.4(x^3-y).$$

Problems

1. For the system

$$2x^2-xy-5x+1=0, \quad x+3\log x-y^2=0$$

find the positive roots with four correct digits.

In Problems 2 and 3 find the roots located in the domain bounded by the straight lines

$$y=0, \quad y=x, \quad x=0.5.$$

2. $2x^2-xy-y^2+2x-2y+6=0$, $y-x-1=0$.

3. $\alpha x^3-y^2-1=0$,

$$xy^3-y-4=0, \quad \alpha=1+0.5\cdot k \ (k=0, 1, \ldots, 5).$$

In Problems 4 and 5 solve the systems of equations. The results must contain five correct digits.

4. $x^2y^2-3x^3-6y^3+8=0,\ x^4-9y+2=0.$

5. $\sin x-y=1.32,\ \cos y-x=-0.85.$

4.3 NEWTON'S METHOD SPREAD TO SYSTEMS OF n EQUATIONS IN n UNKNOWNS

Consider a nonlinear system of equations

$$\left.\begin{aligned} f_1(x_1, x_2, \ldots, x_n)&=0,\\ f_2(x_1, x_2, \ldots, x_n)&=0,\\ \cdots\cdots\cdots&\\ f_n(x_1, x_2, \ldots, x_n)&=0 \end{aligned}\right\} \qquad (1)$$

with real left-hand members.

This system can be rewritten in a more concise form. The totality of the arguments $x_1, x_2, \ldots, x_n$ may be considered as an n-dimensional vector

$$\boldsymbol{x}=\begin{pmatrix} x_1\\ x_2\\ \ldots\\ x_n \end{pmatrix}.$$

Analogously, the totality of functions $f_1, f_2, \ldots, f_n$ is also an n-dimensional vector (vector function)

$$\boldsymbol{f}=\begin{pmatrix} f_1\\ f_2\\ \ldots\\ f_n \end{pmatrix}.$$

Therefore, system (1) is briefly written as:

$$\boldsymbol{f}(\boldsymbol{x})=0. \qquad (2)$$

System (2) is solved by the method of successive approximations.

Suppose the pth approximation

$$\boldsymbol{x}^{(p)}=(x_1^{(p)}, x_2^{(p)}, \ldots, x_n^{(p)})$$

of one of the isolated roots $\boldsymbol{x}=(x_1, x_2, \ldots, x_n)$ of the vector equation (2) is found. Then the exact root of this equation can be represented in the form

$$\boldsymbol{x}=\boldsymbol{x}^{(p)}+\boldsymbol{\varepsilon}^{(p)}, \qquad (3)$$

where $\boldsymbol{\varepsilon}^{(p)}=(\varepsilon_1^{(p)}, \varepsilon_2^{(p)}, \ldots, \varepsilon_n^{(p)})$ is the error of the root (the correction).

Let us introduce into consideration the Jacobi matrix of the system of functions $f_1, f_2, \ldots, f_n$ with respect to the variables $x_1, x_2, \ldots, x_n$:

$$W(x) = \begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} & \cdots & \frac{\partial f_2}{\partial x_n} \\ \cdots & \cdots & \cdots & \cdots \\ \frac{\partial f_n}{\partial x_1} & \frac{\partial f_n}{\partial x_2} & \cdots & \frac{\partial f_n}{\partial x_n} \end{pmatrix} = \left(\frac{\partial f_i}{\partial x_j}\right).$$

If this matrix is nonsingular, i.e. if $\det W(x) \neq 0$, then the correction $\varepsilon^{(p)}$ is expressed in the following way:

$$\varepsilon^{(p)} = -W^{-1}(x^{(p)}) f(x^{(p)}), \tag{4}$$

where $W^{-1}(x^{(p)})$ is the inverse to the Jacobi matrix.

Thus, the successive approximations are found by the formula

$$x^{(p+1)} = x^{(p)} - W^{-1}(x^{(p)}) f(x^{(p)}) \quad (p = 0, 1, 2, \ldots). \tag{5}$$

A rough value of the sought-for root may be taken for the zeroth approximation $x^{(0)}$ (see [12]).

The conditions of convergence of Newton's method have been investigated by L. B. Kantorovich, and A. M. Ostrowski (see [16], [34]).

Example. Using Newton's method, approximate the positive solution of the system of equations

$$x^2 + y^2 + z^2 = 1, \quad 2x^2 + y^2 - 4z = 0, \quad 3x^2 - 4y + z^2 = 0,$$

proceeding from the initial approximation $x_0 = y_0 = z_0 = 0.5$.

Solution. We have

$$f(x) = \begin{pmatrix} x^2 + y^2 + z^2 - 1 \\ 2x^2 + y^2 - 4z \\ 3x^2 - 4y + z^2 \end{pmatrix},$$

whence

$$f(x^{(0)}) = \begin{pmatrix} 0.25 + 0.25 + 0.25 - 1 \\ 0.50 + 0.25 - 2.00 \\ 0.75 - 2.00 + 0.25 \end{pmatrix} = \begin{pmatrix} -0.25 \\ -1.25 \\ -1.00 \end{pmatrix}.$$

Let us form the Jacobi matrix

$$W(x) = \begin{pmatrix} 2x & 2y & 2z \\ 4x & 2y & -4 \\ 6x & -4 & 2z \end{pmatrix}.$$

We have

$$W(x^{(0)}) = \begin{pmatrix} 1 & 1 & 1 \\ 2 & 1 & -4 \\ 3 & -4 & 1 \end{pmatrix}$$

and

$$\det W(x^{(0)}) = \begin{vmatrix} 1 & 1 & 1 \\ 2 & 1 & -4 \\ 3 & -4 & 1 \end{vmatrix} = -40.$$

Find the inverse matrix

$$W^{-1}(x^{(0)}) = -\frac{1}{40}\begin{pmatrix} -15 & -5 & -5 \\ -14 & -2 & 6 \\ -11 & 7 & -1 \end{pmatrix} = \begin{pmatrix} \frac{3}{8} & \frac{1}{8} & \frac{1}{8} \\ \frac{7}{20} & \frac{1}{20} & -\frac{3}{20} \\ \frac{11}{40} & -\frac{7}{40} & \frac{1}{40} \end{pmatrix}.$$

Using formula (5), we get the first approximation

$$x^{(1)} = x^{(0)} - W^{-1}(x^{(0)}) f(x^{(0)}) =$$

$$= \begin{pmatrix} 0.5 \\ 0.5 \\ 0.5 \end{pmatrix} - \begin{pmatrix} \frac{3}{8} & \frac{1}{8} & \frac{1}{8} \\ \frac{7}{20} & \frac{1}{20} & -\frac{3}{20} \\ \frac{11}{40} & -\frac{7}{40} & \frac{1}{40} \end{pmatrix} \begin{pmatrix} -0.25 \\ -1.25 \\ -1.00 \end{pmatrix} =$$

$$= \begin{pmatrix} 0.5 \\ 0.5 \\ 0.5 \end{pmatrix} + \begin{pmatrix} 0.375 \\ 0 \\ -0.125 \end{pmatrix} = \begin{pmatrix} 0.875 \\ 0.500 \\ 0.375 \end{pmatrix}.$$

Then we compute the second approximation $x^{(2)}$. We have

$$f(x^{(1)} = \begin{pmatrix} 0.875^2 + 0.500^2 + 0.375^2 - 1 \\ 2 \cdot 0.875^2 + 0.500^2 - 4 \cdot 0.375 \\ 3 \cdot 0.875^2 - 4 \cdot 0.500 + 0.375^2 \end{pmatrix} = \begin{pmatrix} 0.15625 \\ 0.28125 \\ 0.43750 \end{pmatrix},$$

$$W(x^{(1)}) = \begin{pmatrix} 2 \cdot 0.875 & 2 \cdot 0.500 & 2 \cdot 0.375 \\ 4 \cdot 0.875 & 2 \cdot 0.500 & -4 \\ 6 \cdot 0.875 & -4 & 2 \cdot 0.375 \end{pmatrix} = \begin{pmatrix} 1.750 & 1 & 0.750 \\ 3.500 & 1 & -4 \\ 5.250 & -4 & 0.750 \end{pmatrix}.$$

Hence

$$\det W(x^{(1)}) = \begin{vmatrix} 1.750 & 1 & 0.750 \\ 3.500 & 1 & -4 \\ 5.250 & -4 & 0.750 \end{vmatrix} = \begin{vmatrix} 1.750 & 1 & 0.750 \\ 1.750 & 0 & -4.750 \\ 12.250 & 0 & 3.750 \end{vmatrix} = -64.75,$$

$$W^{-1}(x^{(1)}) = -\frac{1}{64.75}\begin{pmatrix} -15.25 & -3.75 & -4.75 \\ -23.625 & -2.6250 & 9.625 \\ -19.25 & 12.25 & -1.75 \end{pmatrix}.$$

By formula (5) we get

$$\boldsymbol{x}^{(2)} = \boldsymbol{x}^{(1)} - W^{-1}(\boldsymbol{x}^{(1)})\boldsymbol{f}(\boldsymbol{x}^{(1)}) =$$

$$= \begin{pmatrix} 0.875 \\ 0.500 \\ 0.375 \end{pmatrix} + \frac{1}{64.75} \begin{pmatrix} -15.25 & -3.75 & -4.75 \\ -23.625 & -2.6250 & 9.625 \\ -19.25 & 12.25 & -1.75 \end{pmatrix} \begin{pmatrix} 0.15625 \\ 0.28125 \\ 0.43750 \end{pmatrix} =$$

$$= \begin{pmatrix} 0.875 \\ 0.500 \\ 0.375 \end{pmatrix} - \begin{pmatrix} 0.08519 \\ 0.00338 \\ 0.00507 \end{pmatrix} = \begin{pmatrix} 0.78981 \\ 0.49662 \\ 0.36993 \end{pmatrix}.$$

Further approximations are found in a similar way

$$\boldsymbol{x}^{(3)} = \begin{pmatrix} 0.78521 \\ 0.49662 \\ 0.36992 \end{pmatrix}, \quad \boldsymbol{f}(\boldsymbol{x}^{(3)}) = \begin{pmatrix} 0.00001 \\ 0.00004 \\ 0.00005 \end{pmatrix}.$$

Confining ourselves to the third approximation we obtain

$$x = 0.7852, \quad y = 0.4966, \quad z = 0.3699.$$

4.4 APPLYING THE METHOD OF ITERATION TO SYSTEMS OF n EQUATIONS IN n UNKNOWNS

Let there be given a system of nonlinear equations of a special form:

$$\left.\begin{aligned} x_1 &= \varphi_1(x_1, x_2, \ldots, x_n), \\ x_2 &= \varphi_2(x_1, x_2, \ldots, x_n), \\ &\cdots\cdots\cdots\cdots \\ x_n &= \varphi_n(x_1, x_2, \ldots, x_n), \end{aligned}\right\} \tag{1}$$

where the functions $\varphi_1, \varphi_2, \ldots, \varphi_n$ are real, defined and continuous in some neighbourhood ω of an isolated solution $(x_1^*, x_2^*, \ldots, x_n^*)$ of this system, or in a more concise notation:

$$\boldsymbol{x} = \boldsymbol{\varphi}(\boldsymbol{x}), \tag{2}$$

where $\boldsymbol{x} = \begin{pmatrix} x_1 \\ x_2 \\ \ldots \\ x_n \end{pmatrix}$, and $\boldsymbol{\varphi}(\boldsymbol{x}) = \begin{pmatrix} \varphi_1(\boldsymbol{x}) \\ \varphi_2(\boldsymbol{x}) \\ \ldots \\ \varphi_n(\boldsymbol{x}) \end{pmatrix}$.

The vector root is sometimes found by the method of iteration

$$\boldsymbol{x}^{(p+1)} = \boldsymbol{\varphi}(\boldsymbol{x}^{(p)}) \quad (p = 0, 1, 2, \ldots). \tag{3}$$

If a system of equations is given in the general form

$$\boldsymbol{f}(\boldsymbol{x}) = 0, \tag{4}$$

where $f(x)$ is the vector function defined and continuous in the neighbourhood ω of the isolated vector root x^*, then it is written in the equivalent form

$$x = \varphi(x), \tag{5}$$

where $\varphi(x)$ is an *iterative vector function* which is sought for in the form

$$\varphi(x) = x + \Lambda f(x).$$

Matrix Λ is chosen as follows (see the notation on page 102):

$$\Lambda = -W^{-1}(x^{(0)}).$$

We assume here that $W(x^{(0)})$ is a nonsingular matrix.

Substituting $\varphi(x)$ into (3) we get an iterative formula

$$x^{(p+1)} = x^{(p)} + \Lambda f(x^{(p)}). \tag{6}$$

Problems

Solve the following systems of equations using the method of iteration:

1. $\left.\begin{aligned} x &= \log \frac{y}{z} + 1, \\ y &= 0.4 + z^2 - 2x^2, \\ z &= 2 + \frac{xy}{20}, \end{aligned}\right\} x_0 = 1, \quad y_0 = 2.2, \quad z_0 = 2.$

2. $\left.\begin{aligned} x + x^2 - 2yz &= 0.1, \\ y - y^2 + 3xz &= -0.2, \\ z + z^2 + 2xy &= 0.3, \end{aligned}\right\} x_0 = y_0 = z_0 = 0.$

5

THE INTERPOLATION OF FUNCTIONS

5.1 THE STATEMENT OF THE PROBLEM OF INTERPOLATION

Let a function $y=f(x)$ be given by a table:

$$y_0=f(x_0), \quad y_1=f(x_1), \quad \ldots, \quad y_n=f(x_n).$$

The problem of interpolation is usually stated in the following form: *find the polynomial* $P(x)=P_n(x)$ *of degree not higher than* n, *whose values at points* x_i $(i=0, 1, 2, \ldots, n)$ *coincide with the values of the given function, i.e.* $P(x_i)=y_i$.

Geometrically, this means that one has to find an algebraic curve of the form $y=a_0x^n+a_1x^{n-1}+\ldots+a_n$ that passes through the given set of points M_i (x_i, y_i) $(i=0, 1, \ldots, n)$ (Fig. 1).

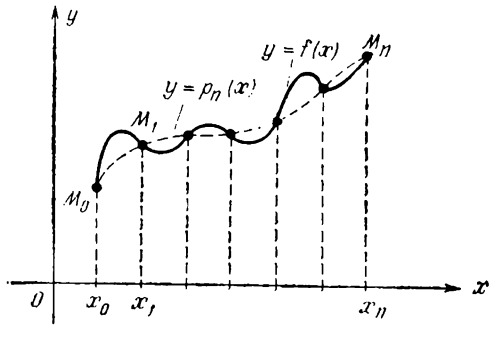

Fig. 1

In such a statement the problem of interpolation is called *parabolic.*

The polynomial $P(x)$ is termed *the interpolation polynomial.* The points x_i $(i=0, \ldots, n)$ are called *mesh points (interpolation points).*

As is proved (see [1], [11], [58]), in the above statement the problem of interpolation always has a unique solution. Interpolation formulas are usually used when finding the unknown values of $f(x)$ for intermediate values of the argument. We distinguish between *the interpolation in the narrow sense*, when the value of x is intermediate between x_0 and x_n, and *extrapolation*, when x is found outside the interval $[x_0, x_n]$.

In estimating the error of the results one must consider both the error of the method of interpolation (the remainder term), and the rounding-off errors of the computations.

5.2 INTERPOLATION FOR THE CASE OF EQUALLY SPACED POINTS. NEWTON'S FIRST AND SECOND INTERPOLATION FORMULAS

Interpolation points are called *equally spaced* if

$$x_{i+1}-x_i=\Delta x_i=h=\text{const} \quad (i=0, 1, \ldots, n-1).$$

The following expressions are called *finite differences* of a function $y=f(x)$:

$\Delta y_i=y_{i+1}-y_i$—finite differences of the first order,
$\Delta^2 y_i=\Delta y_{i+1}-\Delta y_i$—finite differences of the second order,
. . . . . .
$\Delta^k y_i=\Delta^{k-1}y_{i+1}+\Delta^{k-1}y_i$—finite differences of the kth order.

Given below is a horizontal table of finite differences for $n=5$ (see Table 18).

1. *Newton's first interpolation formula* has the form

$$y(x)=P_n(x)= \\ =y_0+q\Delta y_0+\frac{q(q-1)}{2!}\Delta^2 y_0+\ldots+\frac{q(q-1)\ldots(q-n+1)}{n!}\Delta^n y_0, \quad (1)$$

where $q=\frac{x-x_0}{h}$.

Note that the formula uses the upper horizontal row of Table 18. The elements of this row are underlined in the table. The remainder term $R_n(x)$ of formula (1) has the form (see [1], [12])

$$R_n(x)=h^{n+1}\frac{q(q-1)\ldots(q-n)}{(n+1)!}f^{(n+1)}(\xi), \quad (2)$$

where ξ is some internal point of the least interval containing all the points x_i $(i=0, 1, \ldots, n)$ and the point x.

When there is an additional point x_{n+1} we use a more convenient approximate formula for practical computations (see [12]):

$$R_n(x)\approx\frac{\Delta^{n+1}y_0}{(n+1)!}q(q-1)\ldots(q-n). \quad (3)$$

Table 18

Horizontal Table of Finite Differences

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
x_0	y_0	Δy_0	$\Delta^2 y_0$	$\Delta^3 y_0$	$\Delta^4 y_0$
x_1	y_1	Δy_1	$\Delta^2 y_1$	$\Delta^3 y_1$	$\Delta^4 y_1$
x_2	y_2	Δy_2	$\Delta^2 y_2$	$\Delta^3 y_2$	
x_3	y_3	Δy_3	$\Delta^2 y_3$		
x_4	y_4	Δy_4			
x_5	y_5				

The last formula is useful, for example, in the case when a function is given empirically.

The number n should be chosen so that the differences $\Delta^n y_i$ are *practically constant* (see Example 1).

Formula (1) is used for interpolation and extrapolation at points x which are close to the beginning of the table (x_0).

For $n=1$ and $n=2$ we get particular cases:

— *linear interpolation*

$$y(x) = y_0 + q\,\Delta y_0, \tag{4}$$

— *quadratic interpolation*

$$y(x) = y_0 + q\,\Delta y_0 + \frac{q(q-1)}{2!}\,\Delta^2 y_0. \tag{5}$$

Example 1. Given a table of the values of the function $y = \log x$ (Table 19), find $\log 1001$.

Table 19

Values and Differences of the Function $y = \log x$

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	x	y	Δy	$\Delta^2 y$	$\Delta^3 y$
1000	3.0000000	43 214	−426	8	1030	3.0128372	41 961	−401	
1010	3 0043214	42 788	−418	9	1040	3.0170333	41 560		
1020	3.0086002	42 370	−409	8	1050	3.0211893			

Solution. Form a difference table (Table 19), writing the differences to the seventh decimal place, and note that the third differences are practically constant. Therefore, it is sufficient to put $n=3$ in (1):

$$y(x) = y_0 + q\,\Delta y_0 + \frac{q(q-1)}{2!}\,\Delta^2 y_0 + \frac{q(q-1)(q-2)}{3!}\,\Delta^3 y_0.$$

For $x=1001$ we have $q=0.1$. Thus,

$$\log 1001 = 3.0000000 + 0.1 \cdot 0.0043214 + \frac{0.1 \cdot 0.9}{2} 0.0000426 + \\ + \frac{0.1 \cdot 0.9 \cdot 1.9}{6} 0.0000008 = 3.0004341.$$

Estimate the remainder term. By formula (2) for $n=3$ we have

$$R_3(x) = \frac{h^4 q(q-1)(q-2)(q-3)}{4!} f^{(4)}(\xi),$$

where $1000 < \xi < 1030$.

Since $f(x) = \log x$, we have $f^{(4)}(x) = -\frac{3!}{x^4} \log e$; therefore

$$|f^{(4)}(\xi)| < \frac{3!}{(1000)^4} \log e.$$

For $h=10$ and $q=0.1$ we finally get

$$|R_3(1001)| < \frac{0.1 \cdot 0.9 \cdot 1.9 \cdot 2.9 \cdot 10^4 \log e}{4 \cdot (1000)^4} \approx 0.5 \cdot 10^{-9}.$$

Thus, the remainder term can affect only the ninth decimal digit. Note that the obtained value of $\log 1001$ fully coincides with its value in the seven-place table of logarithms.

Example 2. Given the table of the values of the function $y = \frac{1}{x}$ (Table 20), find $\frac{1}{2.718}$ using linear interpolation.

Solution. Determine $\Delta y_0 = -0.0028$ and $q = 0.018/0.02 = 0.9$ and from formula (4) find

$$\frac{1}{2.718} = 0.3704 - 0.0028 \cdot 0.9 = 0.3679.$$

Estimate the remainder term. By formula (2) for $n=1$ we have

$$R_1(x) = \frac{h^2 q(q-1)}{2!} f''(\xi),$$

where $2.70 < \xi < 2.72$.

Since $f(x) = 1/x$, then $f''(x) = 2/x^3$; therefore

$$|R_1(2.718)| < \frac{(0.02)^2 \cdot 0.9 \cdot 0.1}{(2.7)^3} \approx 0.2 \cdot 10^{-5}.$$

Thus, the remainder term can affect only the sixth decimal digit.

Example 3. Using the table of the values of the function $y = e^x$ (Table 21), compute $e^{3.62}$ and $e^{3.58}$ by the formula for quadratic interpolation.

Solution. Compute the differences up to the second order (see Table 21).

Table 20

Values and Differences of the Function $y = 1/x$

x	y	Δy
2.70	0.3704	—28
2.72	0.3676	—26
2.74	0.3650	

Table 21

Values and Differences of the Function $y = e^x$

x	y	Δy	$\Delta^2 y$
3.60	36.598	1.877	95
3.65	38.475	1.972	102
3.70	40.447	2.074	
3.75	42.521		

For $x = 3.62$ we find $q = 0.02/0.05 = 0.4$ and compute by formula (5)

$$e^{3.62} = 36.598 + 0.4 \cdot 1.877 - \frac{0.4 \cdot 0.6}{2} 0.095 = 37.338.$$

For $n = 2$ the remainder term has the form

$$R_2(x) = \frac{h^3 q (q-1)(q-2)}{3!} f'''(\xi).$$

Since $f'''(x) = e^x$ and $3.60 < \xi < 3.70$, we get

$$|R_2(3.62)| < \frac{(0.05)^3 \cdot 0.4 \cdot 0.6 \cdot 1.6}{6} e^{3.70} \approx 0.3 \cdot 10^{-3}.$$

Thus, we may consider all the digits of the answer as correct.

For $x = 3.58$ we find $q = -0.02/0.05 = -0.4$ and determine from formula (5)

$$e^{3.58} = 36.598 - 0.4 \cdot 1.877 + \frac{0.4 \cdot 1.4}{2} 0.095 = 35.874.$$

For estimating the remainder term we have $3.58 < \xi < 3.70$, therefore

$$|R_3(3.58)| < \frac{(0.05)^2 \cdot 0.4 \cdot 1.4 \cdot 2.4}{6} e^{3.70} \approx 10^{-3}.$$

Comparing the remainder terms for $x = 3.62$ and $x = 3.58$, we see that extrapolation for $x = 3.58$ yields a less accurate result.

2. *Newton's second interpolation formula* has the form

$$y(x) = P_n(x) = y_n + q\,\Delta y_{n-1} + \frac{q(q+1)}{2!}\Delta^2 y_{n-2} + \ldots$$
$$\ldots + \frac{q(q+1)\ldots(q+n-1)}{n!}\Delta^n y_0, \quad (6)$$

where $q = \frac{x - x_n}{h}$.

The formula uses the low inclined difference row (see Table 18).

The remainder term $R_n(x)$ of formula (6) has the form (see [1], [14])

$$R_n(x) = h^{n+1} \frac{q(q+1)\ldots(q+n)}{(n+1)!} f^{(n+1)}(\xi), \quad (7)$$

where ξ is an inside point of the least interval containing all the points x_i $(i=0, 1, \ldots, n)$ as well as the point x.

Formula (6) is used for interpolation and extrapolation at points x close to the end of the table, i. e. to x_n.

Example 4. Using the table of the values of the function $y=\sin x$ (see Table 22), find $\sin 54°$ and $\sin 56°$ and indicate the error of the results.

Table 22

Values and Differences of the Function $y=\sin x$

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	x	y	Δy	$\Delta^2 y$	$\Delta^3 y$
30°	0.5000	736	—44	—5	45°	0.7071	589	—57	
35°	0.5736	692	—49	—5	50°	0.7660	532		
40°	0.6428	643	—53	—3	55°	0.8192			

Solution. Forming a difference table (Table 22), we see that the third differences are practically constant. Therefore it is sufficient to take four terms in formula (6).

For computing $\sin 54°$ we have

$$q=\frac{54°-55°}{5°}=-0.2.$$

From formula (6) we get

$$\sin 54°=0.8192+(-0.2)\,0.0532-$$
$$-\frac{(-0.2)\cdot 0.8}{2}\,0.0057-\frac{(-0.2)\cdot 0.8\cdot 1.8}{6}\,0.0003=0.80903.$$

For $n=3$ the remainder term has the form (see formula (7))

$$R_3(x)=h^4\frac{q(q+1)(q+2)(q+3)}{4!}f^{(4)}(\xi).$$

In our case $h=5°=0.0873$, $q=-0.2$, $f^{(4)}(\xi)=\sin\xi\leqslant 1$. Thus,

$$|R_3(54°)|\leqslant\frac{(0.087)^4\cdot 0.2\cdot 0.8\cdot 1.8\cdot 2.8}{24}\approx 0.2\cdot 10^{-5}.$$

We see that the remainder term can affect only the fifth decimal digit; therefore, we write the final result in the form $\sin 54°=0.8090$. The obtained value completely coincides with the one taken from the table.

Now let us find $\sin 56°$. In this case

$$q=\frac{56°-55°}{5°}=0.2,$$

and from (6) we get

$$\sin 56^\circ = 0.8192 + 0.2 \cdot 0.0532 - \frac{0.2 \cdot 1.2}{2} 0.0057 - \frac{0.2 \cdot 1.2 \cdot 2.2}{3!} 0.0003 = 0.82913.$$

For $q = 0.2$, $h = 0.0873$ the remainder term is estimated in the following way:

$$|R_3(56^\circ)| \leqslant \frac{(0.087)^4 \cdot 0.2 \cdot 1.2 \cdot 2.2 \cdot 3.2}{24} \approx 0.4 \cdot 10^{-5}.$$

Hence, the remainder term can affect only the fifth digit. Therefore

$$\sin 56^\circ = 0.8291.$$

Problems

1. Given a table of values of the function $y = e^x$

x	e^x	x	e^x	x	e^x
0.50	1.6487	0.54	1.7160	0.58	1.7860
0.51	1.6653	0.55	1.7333	0.59	1.8040
0.52	1.6820	0.56	1.7507	0.60	1.8221
0.53	1.6989	0.57	1.7683		

Using the formula for linear interpolation, compute e^x for the following values of the argument x and give the estimate for the remainder term R_1.

(a) 0.507, (b) 0.512, (c) 0.523, (d) 0.535, (e) 0.541, (f) 0.556, (g) 0.568, (h) 0.571, (i) o.589, (j) 0.594.

2. Given a table of values of the function $y = \sin x$

x	$\sin x$	x	$\sin x$	x	$\sin x$
1.1	0.89121	1.6	0.99957	2.1	0.86321
1.2	0.93204	1.7	0.99166	2.2	0.80850
1.3	0.96356	1.8	0.97385	2.3	0.74571
1.4	0.98545	1.9	0.94630	2.4	0.67546
1.5	0.99749	2.0	0.90930	2.5	0.59847

Using Newton's first or second formulas for $n = 2$ (quadratic interpolation), compute $\sin x$ for the following values of the argument x and give the estimate for the remainder term R_2.

(a) 1.151, (b) 1.218, (c) 1.345, (d) 1.421, (e) 1.538, (f) 1.609, (g) 1.732, (h) 1.849, (i) 1.929, (j) 2.031, (k) 2.173, (l) 2.218, (m) 2.313, (n) 2.437, (o) 2.478.

3. Given a table of sines with the interval 1′. What is the greatest error of linear interpolation R_1?

4. Given a table of natural logarithms for the numbers from 1 to 10. What is the greatest error of linear interpolation R_1 if the interval is equal to 0.001?

5. Given a table of the values of the probability integral $\frac{2}{\pi}\int_0^x e^{-z^2}dz$ from $x=0$ to $x=3$ spaced at 0.001. What is the greatest error of linear interpolation R_1?

6. Functions $f(x)$, $g(x)$, and $h(x)$ are given by Tables (23), (24), and (25), respectively. Using Newton's first or second interpolation formulas, find the values of these functions for the following values of the argument:

Table 23

x	$f(x)$
1.50	0.51183
1.51	0.50624
1.52	0.50064
1.53	0.49503
1.54	0.48940
1.55	0.48376
1.56	0.47811
1.57	0.47245
1.58	0.46678
1.59	0.46110
1.60	0.45540

Table 24

x	$g(x)$
1.0	0.5652
1.1	0.6375
1.2	0.7147
1.3	0.7973
1.4	0.8861
1.5	0.9817
1.6	1.0848
1.7	1.1964
1.8	1.3172
1.9	1.4482
2.0	1.5906

Table 25

x	$h(x)$
0.00	0.28081
0.05	0.31270
0.10	0.34549
0.15	0.37904
0.20	0.41318
0.25	0.44774
0.30	0.48255
0.35	0.51745
0.40	0.55226
0.45	0.58682
0.50	0.62096

(1) for the function $f(x)$

(a) 1.50911, (b) 1.50820, (c) 1.50253, (d) 1.50192, (e) 1.59513, (f) 1.59575, (g) 1.59614, (h) 1.59728;

(2) for the function $g(x)$

(a) 1.0113, (b) 1.0219, (c) 1.0321, (d) 1.0428, (e) 1.9592, (f) 1.9675, (g) 1.9728, (h) 1.9819;

(3) for the function $h(x)$

(a) 0.01928, (b) 0.01392, (c) 0.02475, (d) 0.02713, (e) 0.47113, (f) 0.47531, (g) 0.48398, (h) 0.48675.

5.3 THE INTERPOLATION FORMULAS OF GAUSS, STIRLING, AND BESSEL

1. The Gaussian interpolation formulas. Gauss' first interpolation formula (for forward interpolation) has the form

$$P(x)=y_0+q\,\Delta y_0+\frac{q(q-1)}{2!}\Delta^2 y_{-1}+\frac{(q+1)\,q(q-1)}{3!}\Delta^3 y_{-1}+$$
$$+\frac{(q+1)\,q(q-1)(q-2)}{4!}\Delta^4 y_{-2}+\frac{(q+2)(q+1)\,q(q-1)(q-2)}{5!}\Delta^5 y_{-2}+\ldots$$
$$\ldots+\frac{(q+n-1)\ldots(q-n+1)}{(2n-1)!}\Delta^{2n-1}y_{-(n-1)}+\frac{(q+n-1)\ldots(q-n)}{(2n)!}\Delta^{2n}y_{-n}, \tag{1}$$

where $q=\frac{x-x_0}{h}$.

The differences Δy_0, $\Delta^2 y_{-1}$, $\Delta^3 y_{-1}$, $\Delta^4 y_{-2}$, $\Delta^5 y_{-2}$, $\Delta^6 y_{-3}$, ..., used in this formula form the lower broken row in Table 26.

Gauss' second interpolation formula (for backward interpolation) is written in the form

$$P(x)=y_0+q\Delta y_{-1}+\frac{(q+1)\,q}{2!}\Delta^2 y_{-1}+\frac{(q+1)\,q(q-1)}{3!}\Delta^3 y_{-2}+$$
$$+\frac{(q+2)(q+1)\,q(q-1)}{4!}\Delta^4 y_{-2}+\ldots+\frac{(q+n-1)\ldots(q-n+1)}{(2n-1)!}\Delta^{2n-1}y_{-n}+$$
$$+\frac{(q+n)(q+n-1)\ldots(q-n+1)}{(2n)!}\Delta^{2n}y_{-n}, \tag{2}$$

where $q=\frac{x-x_0}{h}$.

The differences Δy_{-1}, $\Delta^2 y_{-1}$, $\Delta^3 y_{-2}$, $\Delta^4 y_{-2}$, $\Delta^5 y_{-3}$, $\Delta^6 y_{-3}$, ..., used in this formula, form the upper broken row in Table 26. The remainder term of the formulas (1) and (2) can be written in the form (see [1])

$$R_{2n}=\frac{h^{2n+1}f^{(2n+1)}(\xi)}{(2n+1)!}\,q\,(q^2-1^2)\,(q^2-2^2)\ldots(q^2-n^2), \tag{3}$$

where ξ is an inside point of the interval containing all the points x_i ($i=0, \pm1, \pm2, \ldots, \pm n$) and the point x as well.

The Gaussian formulas are used for interpolation in the middle of the table (close to x_0). The first formula is applied for $x>x_0$, and the second for $x<x_0$.

Example 1. Using the table of the values of the function $y=e^x$ (Table 27), find $e^{1.13}$ and $e^{1.17}$.

Solution. Forming a difference table(Table 27), we note that the third differences are practically constant. Therefore it is sufficient to put $n=3$ in formulas (1) and (2).

Table 26

Diagonal Difference Table

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$	$\Delta^5 y$	$\Delta^6 y$
x_{-4}	y_{-4}						
		Δy_{-4}					
x_{-3}	y_{-3}		$\Delta^2 y_{-4}$				
		Δy_{-3}		$\Delta^3 y_{-4}$			
x_{-2}	y_{-2}		$\Delta^2 y_{-3}$		$\Delta^4 y_{-4}$		
		Δy_{-2}		$\Delta^3 y_{-3}$		$\Delta^5 y_{-4}$	
x_{-1}	y_{-1}		$\Delta^2 y_{-2}$		$\Delta^4 y_{-3}$		$\Delta^6 y_{-4}$
		Δy_{-1}	↓	$\Delta^3 y_{-2}$	↓	$\Delta^5 y_{-3}$	↓
x_0	y_0		$\Delta^2 y_{-1}$	↑ ↓	$\Delta^4 y_{-2}$	↑ ↓	$\Delta^6 y_{-3}$
		Δy_0	↑	$\Delta^3 y_{-1}$	↑	$\Delta^5 y_{-2}$	↑
x_1	y_1		$\Delta^2 y_0$		$\Delta^4 y_{-1}$		$\Delta^6 y_{-2}$
		Δy_1		$\Delta^3 y_0$		$\Delta^5 y_{-1}$	
x_2	y_2		$\Delta^2 y_1$		$\Delta^4 y_0$		
		Δy_2		$\Delta^3 y_1$			
x_3	y_3		$\Delta^2 y_2$				
		Δy_3					
x_4	y_4						

Table 27

Values and Differences of the Function $y = e^x$

i	x_i	y_i	Δy	$\Delta^2 y$	$\Delta^3 y$
−3	1.00	2.7183			
			1394		
−2	1.05	2.8577		71	
			1465		4
−1	1.10	3.0042		75	
			1540		4
0	1.15	3.1582		79	
			1619		4
1	1.20	3.3201		83	
			1702		5
2	1.25	3.4903		88	
			1790		
3	1.30	3.6693			

To compute $e^{1.17}$ we use the formula (1) for $n=3$ and $q=\frac{1.17-1.15}{0.05}=0.4$:

$$e^{1.17}=y_0+q\,\Delta y_0+\frac{q(q-1)}{2!}\Delta^2 y_{-1}+\frac{q(q-1)(q+1)}{3!}\Delta^3 y_{-1}=$$
$$=3.1582+0.4\cdot 0.1619+\frac{0.4\cdot(-0.6)}{2}0.0079+$$
$$+\frac{0.4\cdot(-0.6)\cdot 1.4}{6}0.0004=3.2220.$$

For computing $e^{1.13}$ we use formula (2) for $n=3$ and $q=\frac{1.13-1.15}{0.05}=-0.4$:

$$e^{1.13}=y_0+q\,\Delta y_{-1}+\frac{(q+1)q}{2!}\Delta^2 y_{-1}+\frac{(q+1)q(q-1)}{3!}\Delta^3 y_{-2}=$$
$$=3.1582-0.4\cdot 0.1540-\frac{0.6\cdot 0.4}{2}0.0079+$$
$$+\frac{0.6\cdot(-0.4)\cdot(-1.4)}{6}0.0004=3.0957.$$

2. Stirling's interpolation formula. Taking the arithmetic mean of the first and second interpolation formulas of Gauss, we get this formula

$$P(x)=y_0+q\,\frac{\Delta y_{-1}+\Delta y_0}{2}+\frac{q^2}{2}\Delta^2 y_{-1}+$$
$$+\frac{q(q^2-1^2)}{3!}\,\frac{\Delta^3 y_{-2}+\Delta^3 y_{-1}}{2}+\frac{q^2(q^2-1^2)}{4!}\Delta^4 y_{-2}+$$
$$+\frac{q(q^2-1^2)(q^2-2^2)}{5!}\,\frac{\Delta^5 y_{-3}+\Delta^5 y_{-2}}{2}+\frac{q^2(q^2-1^2)(q^2-2^2)}{6!}\Delta^6 y_{-3}+\dots$$
$$\dots+\frac{q(q^2-1^2)(q^2-2^2)\dots[q^2-(n-1)^2]}{(2n-1)!}\,\frac{\Delta^{2n-1}y_{-n}+\Delta^{2n-1}y_{-(n-1)}}{2}+$$
$$+\frac{q^2(q^2-1^2)\dots[q^2-(n-1)^2]}{(2n)!}\Delta^{2n}y_{-n}, \qquad (4)$$

where $q=\frac{x-x_0}{h}$.

The remainder term of this formula has the same form (3).

The formula is used for interpolation in the middle of the table for the values of q close to zero. In practical applications it is used for $|q|\leqslant 0.25$.

Example 2. Using the table of the values of the function $y=\sinh x$ (Table 28), find sinh 1.41710.

Solution. Form a difference table (Table 28). Noting that the fourth differences are practically constant, we take $n=4$ in formula (4):

$$y(x)=y_0+q\,\frac{\Delta y_{-1}+\Delta y_0}{2}+\frac{q^2}{2}\Delta^2 y_{-1}+$$
$$+\frac{q(q^2-1)}{3!}\,\frac{\Delta^3 y_{-2}+\Delta^3 y_{-1}}{2}+\frac{q^2(q^2-1)}{4}\Delta^4 y_{-2}.$$

Table 28

Values and Differences of the function $y=\sinh x$

i	x_i	y_i	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
−4	1.0	1.17520				
			16 045			
−3	1.1	1.33565		1336		
			17 381		175	
−2	1.2	1.50946		1511		14
			18 892		189	
−1	1.3	1.69838		1700		17
			20 592		206	
0	1.4	1.90430		1906		19
			22 498		225	
1	1.5	2.12928		2131		21
			24 629		246	
2	1.6	2.37557		2377		25
			27 006		271	
3	1.7	2.64563		2648		
			29 654			
4	1.8	2.94217				

For $x=1.41710$ we have

$$q=\frac{1.41710-1.4}{0.1}=0.1710.$$

Thus,

$$\sinh 1.41710=1.90430+0.1710\,\frac{0.20592+0.22498}{2}+$$
$$+\frac{(0.1710)^2}{2}\,0.01906+\frac{0.1710\,((0.1710)^2-1)}{3!}\,\frac{0.00206+0.00225}{2}+$$
$$+\frac{(0.1710)^2\,((0.1710)^2-1)}{4!}\,0.00019=1.90430+0.1710\cdot 0.21545+$$
$$+0.01462\cdot 0.01906-0.02767\cdot 0.00215-0.00118\cdot 0.00019=1.94136.$$

Comparing the obtained result with the corresponding value of the function $y=\sinh x$ taken from the five-place tables, we see that all the digits of the result are correct.

3. Bessel's interpolation formula. This formula has the form

$$P(x)=\frac{y_0+y_{-1}}{2}+\left(q-\frac{1}{2}\right)\Delta y_0+\frac{q(q-1)}{2}\,\frac{\Delta^2 y_{-1}+\Delta^2 y_0}{2}+$$
$$+\frac{(q-0.5)\,q\,(q-1)}{3!}\,\Delta^3 y_{-1}+\frac{q(q-1)(q+1)(q-2)}{4!}\,\frac{\Delta^4 y_{-2}+\Delta^4 y_{-1}}{2}+$$
$$+\frac{(q-0.5)\,q\,(q-1)\,(q+1)\,(q-2)}{5!}\,\Delta^5 y_{-2}+$$
$$+\frac{q\,(q-1)\,(q+1)\,(q-2)\,(q+2)\,(q-3)}{6!}\,\frac{\Delta^6 y_{-3}+\Delta^6 y_{-2}}{2}+\ldots$$

$$\ldots + \frac{q(q-1)(q+1)(q-2)(q+2)\ldots(q-n)(q+n-1)}{(2n)!} \frac{\Delta^{2n}y_{-n} + \Delta^{2n}y_{-n+1}}{2} +$$
$$+ \frac{(q-0.5)\,q(q-1)(q+1)(q-2)(q+2)\ldots(q-n)(q+n-1)}{(2n+1)!} \Delta^{2n+1}y_{-n}, \qquad (5)$$

where $q = \frac{x-x_0}{h}$.

The remainder term can be written in the form (see [1], [14])

$$R_n(x) = \frac{h^{2n+2}}{(2n+2)!} f^{(2n+2)}(\xi)\, q(q^2-1^2)(q^2-2^2)\ldots(q^2-n^2)(q-n-1), \qquad (6)$$

where ξ lies between $x_0 - nh$ and $x_0 + nh$.

Bessel's formula is used for interpolation in the middle of the table for the values of q close to 0.5. In practical applications it is used for $0.25 \leqslant q \leqslant 0.75$. The formula has the simplest form for $q = 0.5$, since all the terms, containing odd differences, disappear. This special case of the Bessel interpolation formula is called the *formula for interpolating to halves*. It is used for forming tables with a smaller interval. For the remainder term for $q = 0.5$ we have (see [14])

$$R_n(x) = \frac{(-1)^{n+1}h^{2n+2}}{(2n+2)!} f^{(2n+2)}(\xi) \frac{[1\cdot 3\cdot 5\ldots(2n+1)]^2}{2^{2n+2}}. \qquad (7)$$

Example 3. Using Table 28, find sinh 1.45224.

Solution. Find

$$q = \frac{1.45224 - 1.4}{0.1} = 0.5224.$$

Since q is close to 0.5, let us make use of Bessel's formula, confining ourselves to the first five terms:

$$y(x) = \frac{y_0 + y_{-1}}{2} + (q-0.5)\,\Delta y_0 + \frac{q(q-1)}{2} \frac{\Delta^2 y_{-1} + \Delta^2 y_0}{2} +$$
$$+ \frac{(q-0.5)\,q(q-1)}{3!} \Delta^3 y_{-1} + \frac{q(q-1)(q+1)(q-2)}{4!} \frac{\Delta^4 y_{-2} + \Delta^4 y_{-1}}{2}.$$

Then we compute separately the expressions

$$\frac{q(q-1)}{2} = -0.12475, \quad \frac{(q-0.5)\,q(q-1)}{3!} = -0.00093,$$
$$\frac{q(q^2-1)(q-2)}{4!} = 0.02339.$$

We find finally

$$\sinh 1.45224 = \frac{1.90430 + 2.12928}{2} + 0.0224 \cdot 0.22498 -$$
$$- 0.12475 \cdot \frac{0.01906 + 0.02131}{2} - 0.00093 \cdot 0.00225 +$$
$$+ 0.02339 \cdot \frac{0.00019 + 0.00021}{2} = 2.01931.$$

Comparing the obtained result with the corresponding value taken from the table, we note that all the digits are correct.

Note. As is obvious from the above considered examples (1 to 3), the terms in the interpolation formulas of Gauss, Stirling, and Bessel decrease considerably more rapidly than those in Newton's formulas, and the last term does not at all affect the accuracy of the final result.

Problems

In problems 1 to 3 make use of the interpolation formulas of Gauss, Stirling, or Bessel.

1. Find the values of the function $f(x)$, given in tabular form (Table 23), for the following values of x:

(a) 1.55031, (b) 1.55193, (c) 1.55232, (d) 1.55381, (e) 1.55482, (f) 1.55568, (g) 1.55632, (h) 1.55713, (i) 1.55823, (j) 1.55912, (k) 1.54113, (l) 1.54228, (m) 1.54343, (n) 1.54435, (o) 1.54581, (p) 1.54612, (q) 1.54641, (r) 1.54718, (s) 1.54813, (t) 1.54921.

2. Find the values of the function $g(x)$ given by Table 24 for the following values of x:

(a) 1.5095, (b) 1.5182, (c) 1.5265, (d) 1.5328, (e) 1.5455, (f) 1.5578, (g) 1.5728, (h) 1.5823, (i) 1.5919, (j) 1.4095, (k) 1.4128, (l) 1.4213, (m) 1.4318, (n) 1.4424, (o) 1.4583, (p) 1.4619, (q) 1.4724, (r) 1.4843, (s) 1.4921.

3. Find the values of the function $h(x)$ given by Table 25 for the following values of x:

(a) 0.25055, (b) 0.25381, (c) 0.26119, (d) 0.26629, (e) 0.26921, (f) 0.27105, (g) 0.27321, (h) 0.28472, (i) 0.28731, (j) 0.29214, (k) 0.20931, (l) 0.20835, (m) 0.21413, (n) 0.21618, (o) 0.22392, (p) 0.22692, (q) 0.23321, (r) 0.23719, (s) 0.24182.

4. Using Bessel's formula, make the tables of values of the function $f(x)$ twice denser:

(a)

x	$f(x)$
0.5	0.9384
0.6	0.9120
0.7	0.8812
0.8	0.8463
0.9	0.8075
1.0	0.7652
1.1	0.7196
1.2	0.6711
1.3	0.6201
1.4	0.5669
1.5	0.5118

(b)

x	$f(x)$
1.0	1.2661
1.1	1.3262
1.2	1.3937
1.3	1.4693
1.4	1.5534
1.5	1.6467
1.6	1.7500
1.7	1.8640
1.8	1.9896
1.9	2.1277
2.0	2.2796

(c)

x	$f(x)$
2.0	1.5906
2.1	1.7455
2.2	1.9141
2.3	2.0978
2.4	2.2981
2.5	2.5167
2.6	2.7554
2.7	3.0161
2.8	3.3011
2.9	3.6126
3.0	3.9534

5.4 LAGRANGE'S INTERPOLATION FORMULA. AITKEN'S SCHEME

Let $x_i\ (i=0, 1, \ldots, n)$ be arbitrarily specified points, and $y_i = f(x_i)$ the values of a function $f(x)$. Then the polynomial of degree n attaining at points x_i the values y_i will be *Lagrange's interpolation polynomial* (see [1], [11], [12], [58]).

$$L_n(x) = \sum_{i=0}^{n} y_i \frac{(x-x_0)(x-x_1)\ldots(x-x_{i-1})(x-x_{i+1})\ldots(x-x_n)}{(x_i-x_0)(x_i-x_1)\ldots(x_i-x_{i-1})(x_i-x_{i+1})\ldots(x_i-x_n)}. \quad (1)$$

The remainder term is

$$R_n(x) = \frac{f^{(n+1)}(\xi)}{(n+1)!}(x-x_0)(x-x_1)\ldots(x-x_n), \quad (2)$$

where ξ is some point of the least interval containing all the interpolation points $x_i\ (i=0, 1, \ldots, n)$ and point x.

The expressions

$$L_i^{(n)}(x) = \frac{(x-x_0)(x-x_1)\ldots(x-x_{i-1})(x-x_{i+1})\ldots(x-x_n)}{(x_i-x_0)(x_i-x_1)\ldots(x_i-x_{i-1})(x_i-x_{i+1})\ldots(x_i-x_n)} \quad (3)$$

are called the *Lagrangian coefficients*.

For computing $L_i^{(n)}(x)$ it is convenient to use the following arrangement of differences, underlining the differences located on the principal diagonal:

$$\left.\begin{array}{cccccc} \underline{x-x_0} & x_0-x_1 & x_0-x_2 & \ldots & x_0-x_n \\ x_1-x_0 & \underline{x-x_1} & x_1-x_2 & & x_1-x_n \\ x_2-x_0 & x_2-x_1 & \underline{x-x_2} & \ldots & x_2-x_n \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ x_n-x_0 & x_n-x_1 & x_n-x_2 & \ldots & \underline{x-x_n} \end{array}\right\}. \quad (4)$$

Let us denote the product of the elements of the ith row by D_i and that of the principal diagonal by $\Pi_{n+1}(x)$, i. e. $\Pi_{n+1}(x) = (x-x_0)(x-x_1)\ldots(x-x_n)$. Then

$$L_i^{(n)}(x) = \frac{\Pi_{n+1}(x)}{D_i} \quad (i=0, 1, \ldots n). \quad (5)$$

To simplify the computational work it is sometimes useful to take advantage of the fact that the form of the Lagrangian coefficients is invariant under an integral linear substitution: if $x = at+b$, $x_j = at_j+b\ (j=0, 1, \ldots, n)$, then

$$L_i^{(n)}(x) = L_i^{(n)}(t).$$

In the case of equally spaced points the problem of interpolation is further simplified when using the tables for Lagrangian coefficients.

If it is required to find not the general expression $L_n(x)$, but only its values for concrete x, and the values of the function are given for sufficiently large number of points, then it is convenient to apply *Aitken's interpolation scheme* (see [1]). According to this scheme the following polynomials are computed in succession

$$L_{i,\,i+1}(x)=\frac{1}{x_{i+1}-x_i}\begin{vmatrix} y_i & x_i-x \\ y_{i+1}x_{i+1}-x \end{vmatrix},$$

$$L_{i,\,i+1,\,i+2}(x)=\frac{1}{x_{i+2}-x_i}\begin{vmatrix} L_{i,\,i+1}(x) & x_i-x \\ L_{i+1,\,i+2}(x) & x_{i+2}-x \end{vmatrix},$$

$$L_{i,\,i+1,\,i+2,\,i+3}(x)=\frac{1}{x_{i+3}-x_i}\begin{vmatrix} L_{i,\,i+1,\,i+2}(x) & x_i-x \\ L_{i+1,\,i+2,\,i+3}(x) & x_{i+3}-x \end{vmatrix}, \text{ etc.}$$

The interpolation polynomial of degree n having at the points x_i the values y_i $(i=0,\ 1,\ 2,\ \ldots,\ n)$, is written in the following way:

$$L_{01\ldots n}(x)=\frac{1}{x_n-x_0}\begin{vmatrix} L_{01\ldots(n-1)}(x) & x_0-x \\ L_{12\ldots n}(x) & x_n-x \end{vmatrix}.$$

For the sake of convenience the above findings are usually tabulated.

Table 29

Aitken's Interpolation Scheme

x_i	y_i	x_i-x	$L_{i-1,\,i}$	$L_{i-2,\,i-1,\,i}$	$L_{i-3,\,i-2,\,i-1,\,i}$	...
x_0	y_0	x_0-x				
x_1	y_1	x_1-x	$L_{01}(x)$			
x_2	y_2	x_2-x	$L_{12}(x)$	$L_{012}(x)$		
x_3	y_3	x_3-x	$L_{23}(x)$	$L_{123}(x)$	$L_{0123}(x)$	...
x_4	y_4	x_4-x	$L_{34}(x)$	$L_{234}(x)$	$L_{1234}(x)$	...

In case of Aitken's scheme the computations are carried out until the successive values $L_{01\ldots n}(x)$ and $L_{01\ldots n,\,(n+1)}(x)$ coincide within the limits of the specified accuracy. The scheme is readily realized on a computer and ensures the possibility of automatically checking the accuracy of computations.

Example 1. Write the Lagrangian interpolation polynomial for the function $f(x)$ given in tabular form

i	0	1	2	3
x_i	0	0.1	0.3	0.5
y_i	−0.5	0	0.2	1

Solution. By formula (3) for $n=3$ we get the expressions $L_i^{(3)}(x)$ at $i=0, 2, 3$:

$$L_0^{(3)}(x)=\frac{(x-0.1)(x-0.3)(x-0.5)}{(-0.1)(-0.3)(-0.5)}=-\frac{x^3-0.9x^2+0.23x-0.015}{0.015},$$

$$L_2^{(3)}(x)=\frac{x(x-0.1)(x-0.5)}{0.3\cdot 0.2(-0.2)}=-\frac{x^3-0.6x^2-0.05x}{0.012},$$

$$L_3^{(3)}(x)=\frac{x(x-0.1)(x-0.3)}{0.5\cdot 0.4\cdot 0.2}=\frac{x^3-0.4x^2+0.03x}{0.04}$$

(there is no need here to compute $L_1^{(3)}(x)$, since $y_1=0$).

Then the required polynomial will have the form

$$L_3(x)=L_0^{(3)}(x)\,y_0+L_1^{(3)}(x)\,y_1+L_2^{(3)}(x)\,y_2+L_3^{(3)}(x)\,y_3=$$

$$=\frac{125}{3}x^3-30x^2+\frac{73}{12}x-0.5.$$

Example 2. The function $y=f(x)$ is given by the table

x	0.05	0.15	0.20	0.25	0.35	0.40	0.50	0.55
y	0.9512	0.8607	0.8187	0.7788	0.7047	0.6703	0.6065	0.5769

Find $f(0.45)$.

Solution. For the sake of simplicity we put

$$x=0.05\cdot t.$$

Then the values of the new variable t corresponding to the interpolation points will be 1, 3, 4, 5, 7, 8, 10, 11. Besides for $x=0.45$ we have $t=9$.

Taking advantage of the invariance of the Lagrangian coefficients, let us compute $L_i^{(n)}(t)$ instead of $L_i^{(n)}(x)$ (see a table below).

Thus, we get

$$y(0.45)=\Pi(t)\sum_{i=0}^{7}\frac{y_i}{D_i}=\Pi(t)\cdot S=3840\cdot 1.6604\cdot 10^{-4}=0.6376.$$

Example 3. To what accuracy is it possible to compute ln 100.5 by Lagrange's formula, given the values of ln 100, ln 101, ln 102, ln 103?

Solution. The remainder term of Lagrange's formula for $n=3$ has the form

$$R_3(x)=\frac{f^{(4)}(\xi)}{4!}(x-x_0)(x-x_1)(x-x_2)(x-x_3).$$

Table 30

Computing Lagrangian Coefficients

i	$t_i - t_j$ $(i \neq j)$							D_i	y_i	$\frac{y_i}{D_i}$
0	8	−2	−3	−4	−6	−7	−8	−10	−725 760	0.9512	$-0.0131 \cdot 10^{-4}$
1	2	6	−1	−2	−4	−5	−7	−8	26 880	0.8607	$0.3202 \cdot 10^{-4}$
2	3	1	5	−1	−3	−4	−6	−7	−7560	0.8187	$-1.0829 \cdot 10^{-4}$
3	4	2	1	4	−2	−3	−5	−6	5760	0.7788	$1.3520 \cdot 10^{-4}$
4	6	4	3	2	2	−1	−3	−4	−3456	0.7047	$-2.0390 \cdot 10^{-4}$
5	7	5	4	3	1	1	−2	−3	2520	0.6703	$2.6599 \cdot 10^{-4}$
6	9	7	6	5	3	2	−1	−1	11 340	0.6065	$0.5348 \cdot 10^{-4}$
7	10	8	7	6	4	3	1	−2	−80 640	0.5769	$-0.0715 \cdot 10^{-4}$
$\Pi(t) = 3840$									$S = \sum_{i=0}^{7} \frac{y_i}{D_i} = 1.6604 \cdot 10^{-4}$		

In our case $x_0 = 100$, $x_1 = 101$, $x_2 = 102$, $x_3 = 103$, $x = 100.5$; $100 < \xi < 103$. Since $f(x) = \ln x$, then $f^{(4)}(x) = -6/x^4$. Hence,

$$|R_3(100.5)| \leqslant \frac{6}{(100)^4 \cdot 4!} 0.5 \cdot 0.5 \cdot 1.5 \cdot 2.5 = 0.23 \cdot 10^{-8}.$$

Example 4. The function $y = \sqrt[3]{x}$ is given in tabular form:

x	1.0	1.1	1.3	1.5	1.6
y	1.000	1.032	1.091	1.145	1.170

Making use of Aitken's scheme, find $\sqrt[3]{1.15}$.

Solution. Tabulate the given values of the function and compute the differences $x_i - x$ for $x = 1.15$. Then successively find:

$$L_{01} = \frac{1}{x_1 - x_0} \begin{vmatrix} y_0 & x_0 - x \\ y_1 & x_1 - x \end{vmatrix} = \frac{1}{0.1} \begin{vmatrix} 1 & -0.15 \\ 1.032 & -0.05 \end{vmatrix} = 1.048,$$

$$L_{12} = \frac{1}{x_2 - x_1} \begin{vmatrix} y_1 & x_1 - x \\ y_2 & x_2 - x \end{vmatrix} = \frac{1}{0.2} \begin{vmatrix} 1.032 & -0.05 \\ 1.091 & 0.15 \end{vmatrix} = 1.047,$$

$$L_{23} = \frac{1}{x_3 - x_2} \begin{vmatrix} y_2 & x_2 - x \\ y_3 & x_3 - x \end{vmatrix} = \frac{1}{0.2} \begin{vmatrix} 1.091 & 0.15 \\ 1.145 & 0.35 \end{vmatrix} = 1.050,$$

$$L_{34} = \frac{1}{x_4 - x_3} \begin{vmatrix} y_3 & x_3 - x \\ y_4 & x_4 - x \end{vmatrix} = \frac{1}{0.1} \begin{vmatrix} 1.145 & 0.35 \\ 1.170 & 0.45 \end{vmatrix} = 1.057.$$

Table 31

Aitken's Scheme for Example 4

i	x	y	x_i-x	$L_{i-1,\,i}$	$L_{i-2,\,i-1,\,i}$	i	x	y	x_i-x	$L_{i-1,\,i}$	$L_{i-2,\,i-1,\,i}$
0	1.0	1.000	—0.15			3	1.5	1.145	0.35	1.050	
1	1.1	1.032	—0.05	1.048		4	1.6	1.170	0.45	1.057	
2	1.3	1.091	0.15	1.047	1.048						

Enter the obtained values in Table 31 and then compute

$$L_{012}=\frac{1}{0.3}\begin{vmatrix}1.048 & -0.15\\ 1.047 & 0.15\end{vmatrix}=1.048.$$

The values L_{01} and L_{012} coincide to the third digit. Here we stop; hence

$$\sqrt[3]{1.15}=1.048$$

to within 10^{-3}.

Problems

1. Find the polynomial of the least degree which attains the preset values at the given points:

(a)

x	y
1.45	3.14
1.36	4.15
1.14	5.65

(b)

x	y
0	2
1	3
2	12
5	147

(c)

x	y
0	1.45
1.5	3.14
3.4	4.65
6.8	4.11

(d)

x	y
11	1342
13	2210
14	2758
18	5850
19	6878
21	9282

(e)

x	y
—2	25
1	—8
2	—15
4	—23

(f)

x	y
—2	6
—1	0
0	2
1	0
2	6

2. Given a table of the values of the function

x	1.50	1.54	1.56	1.60	1.63	1.70
y	3.873	3.924	3.950	4.000	4.037	4.123

Using Lagrange's formula, find the values of the function at the indicated points:

(a) 1.52, (b) 1.55, (c) 1.58, (d) 1.61, (e) 1.67.

3. Given a table of the values of the function

x	2.0	2.3	2.5	3.0	3.5	3.8	4.0
y	5.848	6.127	6.300	6.694	7.047	7.243	7.368

Using Lagrange's formula, find the values of the function at the indicated points:

(a) 2.22, (b) 2.41, (c) 2.78, (d) 3.34, (e) 3.75, (f) 3.88.

4. Knowing the values of $\sin x$ for $x=0, \frac{\pi}{6}, \frac{\pi}{4}, \frac{\pi}{3}, \frac{\pi}{2}$, find $\sin x$ for $x=\frac{\pi}{12}$ and estimate the error.

5. Knowing the values of $\cos x$ for $x=0, \frac{\pi}{6}, \frac{\pi}{4}, \frac{\pi}{3}, \frac{\pi}{2}$, find $\cos x$ for $x=\frac{\pi}{5}$ and estimate the error.

6. Using Aitken's scheme, find the values of the functions specified in tabular form at the given points:

(a)

x	14	17	31	35
y	68.7	64.0	44.0	39.1

Find y (27).

(b)

x	93.0	96.2	100.0	104.2	108.7
y	11.38	12.80	14.70	17.07	19.91

Find y (102).

(c)

x	0	2	3	6	7	9
y	658 503	704 969	729 000	804 357	830 584	884 736

Find y (5).

7. The function $f(x)$ is given in tabular form

x	1.00	1.08	1.13	1.20	1.27	1.31	1.38
$f(x)$	1.17520	1.30254	1.38631	1.50946	1.21730	1.22361	1.23470

Using Aitken's interpolation scheme, compute to within 10^{-5} the values of the above function for the following values of x: (a) 1.134, (b) 1.139, (c) 1.143, (d) 1.151, (e) 1.166, (f) 1.175, (g) 1.182, (h) 1.197, (i) 1.185, (j) 1.192, (k) 1.195, (l) 1.178.

5.5 INVERSE INTERPOLATION

Suppose we have a function $y=f(x)$ given in tabular form.

The problem of inverse interpolation consists in determining a value of the argument x from a given value of the function y. We assume that the function $f(x)$ is *monotonic* in the interval under consideration, hence the problem in question has a unique solution. In this case the problem is solved directly by means of Lagrange's interpolation polynomial. To do so, it suffices to take the variable y as the independent variable and to write a formula expressing x as a function of y

$$x=\sum_{i=0}^{n} x_i \frac{(y-y_0)(y-y_1)\dots(y-y_{i-1})(y-y_{i+1})\dots(y-y_n)}{(y_i-y_0)(y_i-y_1)\dots(y_i-y_{i-1})(y_i-y_{i+1})\dots(y_i-y_n)}, \quad (1)$$

where $y_i=f(x_i)$ $(i=0, 1, \dots, n)$. Here the remainder term can be obtained from the remainder term of Lagrange's formula by changing the places of x and y.

Example 1. The function $y=f(x)$ is given in tabular form

x	10	15	17	20
y	3	7	11	17

Find the value of x corresponding to $y=10$.

Solution. In this case Lagrange's interpolation polynomial has the form

$$x(y)=\sum_{i=0}^{3} x_i L_i^{(3)}(y),$$

where $L_i^{(3)}(y)$ are the Lagrangian coefficients

For $y=10$ we get

$$x(10)=10\frac{(10-7)(10-11)(10-17)}{(3-7)(3-11)(3-17)}+15\frac{(10-3)(10-11)(10-17)}{(7-3)(7-11)(7-17)}+$$
$$+17\frac{(10-3)(10-7)(10-17)}{(11-3)(11-7)(11-17)}+20\frac{(10-3)(10-7)(10-11)}{(17-3)(17-7)(17-11)}=$$
$$=-\frac{15}{32}+\frac{147}{32}+\frac{17\cdot 49}{64}-\frac{1}{2}=\frac{33}{8}+\frac{833}{64}-\frac{1}{2}=16.641.$$

In this case, however, Aitken's scheme (see Sec. 5.4) would be much more convenient.

Iterative methods. If a function $y=f(x)$ is given by a table with equally spaced points, then we write for it one of the interpolation polynomials, for instance, Newton's first interpolation polynomial:

$$y=y_0+\frac{\Delta y_0}{1!}q+\frac{\Delta^2 y_0}{2!}q(q-1)+\ldots+\frac{\Delta^n y_0}{n!}q(q-1)\ldots(q-n+1). \quad (2)$$

Considering the last expression as a function with respect to q, find q by means of the given value of y, and then compute

$$x=x_0+qh.$$

If the number of points is great, then we obtain an algebraic equation of a high degree, which is conveniently solved by the method of iteration. For this purpose rewrite equation (2) as

$$q=\varphi(q)=\frac{y-y_0}{\Delta y_0}-\frac{\Delta^2 y_0}{2!\,\Delta y_0}q(q-1)-\ldots-$$
$$-\frac{\Delta^n y_0}{n!\,\Delta y_0}q(q-1)\ldots(q-n+1).$$

We take for the initial approximation

$$q_0=\frac{y-y_0}{\Delta y_0},$$

and then use the iteration process

$$q_m=\varphi(q_{m-1}).$$

In most cases the process of iteration converges for a sufficiently small spacing $h=x_{i+1}-x_i$, and it converges to the required root, i.e.

$$q=\lim_{m\to\infty}q_m.$$

The condition of convergence is the fulfillment of the inequality $|\varphi'(q)|\leqslant\alpha<1$. Actually, the process of iteration is continued until two successive values q_m and q_{m+1} coincide with the required accuracy, and it is assumed that $q\approx q_{m+1}$.

Note. The considered method of iteration for inverse interpolation requires that the function $f(x)$ be monotonic only on the interval (x_0, x_1), where $y_0 < y < y_1$, or $y_0 > y > y_1$. However, the points of the used interpolation formula can lie as well beyond the interval of monotonicity of the function $f(x)$.

Example 2. Using the table of the values of the function $y = \sinh x$ (see Table 32), find x for which $\sinh x = 5$.

Table 32

Values of the Function $y = \sinh x$

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$
2.2	4.457	1.009	0.220	0.054
2.4	5.466	1.229	0.274	0.043
2.6	6.695	1.503	0.317	
2.8	8.198	1.820		
3.0	10.018			

Solution. Let us form Newton's first interpolation polynomial, confining ourselves to the third-order differences, which are practically already constant:

$$y = y_0 + \Delta y_0 q + \frac{\Delta^2 y_0}{2} q(q-1) + \frac{\Delta^3 y_0}{6} q(q-1)(q-2).$$

We put $x_0 = 2.2$, since the given value $y = 5$ is found between $y_0 = 4.457$ and $y_1 = 5.466$.

In our case the iterating function has the form

$$\varphi(q) = \frac{y - y_0}{\Delta y_0} - \frac{\Delta^2 y_0}{2\Delta y_0} q(q-1) - \frac{\Delta^3 y_0}{6\Delta y_0} q(q-1)(q-2).$$

Assuming $y_0 = 4.457$, we determine the initial approximation

$$q_0 = \frac{5 - 4.457}{1.009} = \frac{0.543}{1.009} = 0.538.$$

Then we consecutively find

$$q_1 = \frac{y - y_0}{\Delta y_0} - \frac{\Delta^2 y_0}{2\Delta y_0} q_0(q_0-1) - \frac{\Delta^3 y_0}{6\Delta y_0} q_0(q_0-1)(q_0-2) =$$

$$= 0.538 + \frac{0.538 \cdot 0.462}{2} \cdot \frac{0.220}{1.009} - \frac{0.538 \cdot 0.462 \cdot 1.462}{6} \cdot \frac{0.054}{1.009} = 0.5641,$$

$$q_2 = \frac{y - y_0}{\Delta y_0} - \frac{\Delta^2 y_0}{2\Delta y_0} q_1(q_1-1) - \frac{\Delta^3 y_0}{6\Delta y_0} q_1(q_1-1)(q_1-2) =$$

$$= 0.538 + \frac{0.5644 \cdot 0.4356}{2} \cdot \frac{0.220}{1.009} - \frac{0.5644 \cdot 0.4356 \cdot 1.4356}{6} \cdot \frac{0.054}{1.009} = 0.5644.$$

Thus, we may take $q=0.564$ and

$$x=2.2+0.564\cdot 0.2=2.2+0.113=2.313$$

correct to three decimal places.

Example 3. Given a table of values of the probability integral (Table 33)

Table 33

Values of the Probability Integral

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
0.45	0.4754818				
		91 737			
0.46	0.4846555		—840	—11	
		90 897			1
0.47	0.4937452		—851		
		90 046		—10	
0.48	0.5027498		—861		2
		89 185		—8	
0.49	0.5116683		—869		
		88 316			
0.50	0.5204999				

$$y=\frac{2}{\sqrt{\pi}}\int_0^x e^{-z^2}\,dz.$$

Determine the value of x for which the integral equals $\frac{1}{2}$.

Solution. As is seen from the table, the nearest least value of the argument x corresponding to the value of the function $y=\frac{1}{2}$ is $x_0=0.47$. Therefore, here it is convenient to use Bessel's formula

$$y=\frac{y_0+y_1}{2}+\left(q-\frac{1}{2}\right)\Delta y_0+\frac{q(q-1)}{2}\,\frac{\Delta^2 y_{-1}+\Delta^2 y_0}{2}+$$
$$+\frac{(q-1/2)\,q\,(q-1)}{3!}\Delta^3 y_{-1}. \quad (3)$$

In our case $x_0=0.47$, $h=0.01$, $y=0.5$.

For the sake of simplicity we substitute $p=q-0{,}5$, whence $q=p+0.5$. Then formula (3) takes the form

$$0.5=\frac{y_0+y_1}{2}+p\,\Delta y_0+\frac{p^2-0.25}{2}\,\frac{\Delta^2 y_{-1}+\Delta^2 y_0}{2}+\frac{p\,(p^2-0.25)}{3!}\Delta^3 y_{-1}.$$

Separating the term containing p, we get

$$p=\frac{1}{\Delta y_0}\left(-\frac{y_0+y_1}{2}+0.5\right)-\frac{p^2-0.25}{2}\,\frac{\Delta^2 y_{-1}+\Delta^2 y_0}{2\Delta y_0}-$$
$$-\frac{p\,(p^2-0.25)}{6}\,\frac{\Delta^3 y_{-1}}{\Delta y_0}=\varphi(p).$$

For the initial approximation we take

$$p_0 = \frac{1}{\Delta y_0}\left(0.5 - \frac{y_0 + y_1}{2}\right) =$$

$$= \frac{1}{0.0090046}\left(0.5 - \frac{0.4937452 + 0.5027448}{2}\right) = 0.194623.$$

Then we consecutively find

$$p_1 = p_0 - \frac{p_0^2 - 0.25}{2}\,\frac{\Delta^2 y_{-1} + \Delta^2 y_0}{2\Delta y_0} - \frac{p_0(p_0^2 - 0.25)}{6}\,\frac{\Delta^3 y_{-1}}{\Delta y_0} =$$

$$= 0.194623 + 4.753 \cdot 10^{-3}\,[(0.194623)^2 - 0.25] +$$

$$+ 1.85 \cdot 10^{-5} \cdot 0.194623\,[(0.194623)^2 - 0.25] \cdot (-0.0000185) =$$

$$= 0.194623 - 0.00108 - 0.000001 = 0.193614,$$

$$p_2 = p_0 - \frac{p_1^2 - 0.25}{2}\,\frac{\Delta^2 y_{-1} + \Delta^2 y_0}{2\Delta y_0} - \frac{p_1(p_1^2 - 0.25)}{6}\,\frac{\Delta^3 y_{-1}}{\Delta y_0} = 0.193612.$$

Thus, we can write (correct to five decimal places)

$$p \approx 0.19361,$$

whence

$$q = p + 0.5 = 0.69361, \quad x = x_0 + qh = 0.4769361.$$

The required value of x is determined accurate to seven correct digits.

Problems

1. Using the tables of values of functions, determine the value of the argument x corresponding to the given values of y

(a)

x	1	2	2.5	3
y	−6	−1	5.625	16

, $y = 0$,

b)

x	4	6	8	10
y	11	27	50	83

, $y = 20$.

Given in Problems 2 to 4 are tables of the values of monotonic functions. Applying one of the above considered methods of inverse interpolation, find the values of x corresponding to the indicated values of y.

2.

x	1.00	1.05	1.10	1.15	1.20	1.25
y	2.00000	2.00238	2.00909	2.01957	2.03313	2.05000

(a) 2.00139, (b) 2.00194, (c) 2.00373, (d) 2.00484, (e) 2.01147, (f) 2.01452, (g) 2.02259, (h) 2.02888, (i) 2.03805, (j) 2.04610.

3.

x	0.50	0.55	0.60	0.65	0.70
y	2.25000	2.12068	2.02667	1.96096	1.91857

(a) 2.18038, (b) 2.14111, (c) 2.08318, (d) 2.04815, (e) 1.99327, (f) 1.98295, (g) 1.94323, (h) 1.93379.

4.

x	1.00	1.10	1.15	1.20	1.23	1.30
y	1.36788	1.24196	1.18621	1.13452	1.10530	1.04176

(a) 1.27153, (b) 1.21577, (c) 1.19919, (d) 1.17661, (e) 1.15988, (f) 1.11684, (g) 1.11461, (h) 1.08836, (i) 1.06913.

Tabulated in Problems 5 to 7 are the values of non-monotone functions. Using one of the interpolation formulas find the values of x corresponding to the given values of y.

5.

x	0.05	0.10	0.15	0.20
y	0.52943	0.94147	1.14749	1.10930

(a) 0.57769, (b) 0.81315, (c) 1.06549.

6.

x	1.10	1.15	1.20	1.25	1.30	1.35
y	0.91909	0.89207	0.87333	0.86250	0.85923	0.86324

(a) 0.91124, (b) 0.89777, (c) 0.88479, (d) 0.87484, (e) 0.86784, (f) 0.86371.

7.

x	0.05	0.10	0.15	0.20
y	0.48193	0.85147	1.01999	0.94930

(a) 0.73787, (b) 0.79778, (c) 0.91718.

5.6 FINDING THE ROOTS OF AN EQUATION BY INVERSE INTERPOLATION

Suppose we have to solve the equation

$$f(x)=0.$$

Consider the function $y=f(x)$ and form a table of its values close to zero, choosing the number of points depending on the required accuracy of the root. Take the neighbouring points at which

$$f(x_0)\,f(x_1)<0$$

as x_0 and x_1, and, using the method of inverse interpolation, find the value of x for which $y=0$. Here Aitken's scheme is also preferable.

Example 1. Solve the equation

$$e^{-x}-x=0.$$

Solution. This equation has obviously only one root which lies in the interval (0.5; 0.7). Form a table of the values of the function $y=e^{-x}-x$ with spacing 0.05 for the indicated interval:

x	e^{-x}	$y=e^{-x}-x$	x	e^{-x}	$y=e^{-x}-x$
0.50	0.60653	0.10653	0.65	0.52205	—0.12795
0.55	0.57695	0.02695	0.70	0.49658	—0.20342
0.60	0.54881	—0.05119			

As is seen from the table, y alters the sign when passing from the point $x=0.55$ to the point $x=0.60$. Put $x_0=0.55$, $x_1=0.60$ and carry out the computations by Aitken's scheme.

The results of the computations are tabulated below:

i	$y_i=e^{-x_i}-x_i$	$0-y_i$	x_i	$P_{0,i}(0)$	$P_{0,1,i}(0)$
0	0.02695	—0.02695	0.55		
1	—0.05119	0.05119	0.60	0.567245	
2	0.10653	—0.10653	0.50	0.566933	0.567144
3	—0.12795	0.12795	0.65	0.567398	0.567142

The last two approximate values of the root, i.e. 0.567144 and 0.567142, have the same signs in all places to 10^{-5}. Therefore, $x=0.56714$ may be taken as the solution of the problem with an error not more than 10^{-5}.

Problems

In Problems 1 to 15 with the aid of inverse interpolation find (with the preassigned accuracy ε) the root of the equation lying in the interval $[a, b]$.

1. $x^2+\ln x=0$, $\varepsilon=10^{-4}$, $a=0.5$, $b=1$.
2. $x^2-\log(x+2)=0$, $\varepsilon=10^{-4}$, $a=0.5$, $b=1$.
3. $x^2+\ln x-4=0$, $\varepsilon=10^{-4}$, $a=1.5$, $b=2$.
4. $(x-1)^2-0.5e^x=0$, $\varepsilon=10^{-5}$, $a=0.2$, $b=0.3$.
5. $(x-1)^2-e^{-x}=0$, $\varepsilon=10^{-5}$, $a=1.4$, $b=1.5$.
6. $x^3-\sin x=0$, $\varepsilon=10^{-4}$, $a=0$, $b=1$.
7. $4x-\cos x=0$, $\varepsilon=10^{-4}$, $a=0$, $b=0.5$.
8. $x^2-\sin x=0$, $\varepsilon=10^{-4}$, $a=0.5$, $b=1$.
9. $x-\cos x=0$, $\varepsilon=10^{-4}$, $a=0.5$, $b=1$.
10. $x^2-\cos \pi x=0$, $\varepsilon=10^{-4}$, $a=0$, $b=0.5$.
11. $2\sqrt{x}-\cos(\pi x/2)=0$, $\varepsilon=10^{-5}$, $a=0.2$, $b=0.3$.
12. $\sqrt{x}-2\cos(\pi x/2)=0$, $\varepsilon=10^{-4}$, $a=0.7$, $b=0.8$.
13. $x^2-\cot(\pi x/3)=0$, $\varepsilon=10^{-5}$, $a=0.8$, $b=0.9$.
14. $x^2-\cos^2 \pi x=0$, $\varepsilon=10^{-5}$, $a=0.3$, $b=0.4$.
15. $x^2-\sin \pi x=0$, $\varepsilon=10^{-4}$, $a=0.75$, $b=0.85$.
16. Find the least positive root of the equation $\tan mx-ax=0$ to within $\varepsilon=10^{-3}$ for the following values of the parameters:

Variant No.	1	2	3	4	5	6	7	8	9	10
m	1.1	1.2	1.3	1.4	1.5	1.6	1.7	1.8	1.9	2.0
a	2	3	4	5	6	2	3	4	5	6

17. Find the least positive root of the equation

$$\cot mx - ax = 0$$

to within 10^{-4} for the following values of the parameters:

Variant No.	1	2	3	4	5	6	7	8	9	10
m	1.1	1.2	1.3	1.4	1.5	1.6	1.7	1.8	1.9	2.0
a	2	3	4	5	2	3	4	5	6	7

18. Find the least positive root of the equation

$$\sin kx - bx = 0$$

to within 10^{-4} for the following values of the parameters:

Variant No.	1	2	3	4	5	6	7	8	9	10
b	1.1	1.2	1.3	1.4	1.5	1.6	1.7	1.8	2.0	2.1
k	2	2	2	2	2	3	3	3	3	3

In Problems 19 to 27 find the roots of algebraic roots lying on the interval $[a, b]$ to within 10^{-4}.

19. $x^4 + 3x^3 - 9x - 9 = 0$, $a = 1$, $b = 2$.
20. $x^4 - 4x^3 + 4x^2 - 4 = 0$, $a = -1$, $b = 0$.
21. $x^4 + 3x^3 + 4x^2 + x - 3 = 0$, $a = 0$, $b = 1$.
22. $x^4 - 10x^2 - 16x + 5 = 0$, $a = 0$, $b = 1$.
23. $x^4 - x^3 - 9x^2 + 10x - 10 = 0$, $a = -4$, $b = -3$.
24. $x^4 - 6x^2 + 12x - 8 = 0$, $a = 1$, $b = 2$.
25. $x^4 - 3x^2 + 4x - 3 = 0$, $a = -3$, $b = -2$.
26. $x^4 - x^3 - 7x^2 - 8x - 6 = 0$, $a = 3$, $b = 4$.
27. $x^4 - 3x^3 + 3x^2 - 2 = 0$, $a = 1$, $b = 2$.

6

NUMERICAL DIFFERENTIATION

6.1 FORMULAS FOR NUMERICAL DIFFERENTIATION

The below considered formulas for numerical differentiation are used in the cases when a function $y=f(x)$ is represented by a table of its values $y_i=f(x_i)$ for equally spaced points $x_i=x_0+ih$ $(i=0, \pm 1, \pm 2, \ldots)$. Choosing a set of $n+1$ points, replace the function $y=f(x)$ by an interpolation polynomial $P_n(x)$. Then the derivative of this polynomial $P_n'(x)$ is used for numerical differentiation of the required derivative $y'=f'(x)$:

$$f'(x) \approx P_n'(x).$$

For instance, if we choose the points x_0, x_1, x_2, x_3, x_4 and take advantage of Newton's first interpolation polynomial (see Sec. 5.2), then we get the formula for numerical differentiation of the form

$$f'(x) \approx \frac{1}{h}\left(\Delta y_0 + \frac{2q-1}{2}\Delta^2 y_0 + \frac{3q^2-6q+2}{6}\Delta^3 y_0 + \right.$$
$$\left. + \frac{2q^3-9q^2+11q-3}{12}\Delta^4 y_0\right), \qquad (1)$$

where $q=\dfrac{x-x_0}{h}$.

If we take the interpolation points x_{-2}, x_{-1}, x_0, x_1, x_2 and use Stirling's interpolation polynomial (see Sec. 5.3), we get

$$f'(x) \approx \frac{1}{h}\left(\frac{\Delta y_{-1}+\Delta y_0}{2} + q\,\Delta^2 y_{-1} + \frac{3q^2-1}{6}\cdot\frac{\Delta^3 y_{-2}+\Delta^3 y_{-1}}{2} + \right.$$
$$\left. + \frac{2q^3-q}{12}\Delta^4 y_{-2}\right), \qquad (2)$$

where $q=\dfrac{x-x_0}{h}$.

Formulas for numerical differentiation are usually used for finding derivatives at points x_i. Since in this case any point may be taken for the initial one, all formulas are written for the point x_0, which is equivalent to substituting the value $q=0$ into

the formula of the form (1) or (2). Then differentiation of Newton's interpolation polynomials leads to the following formulas:

$$y_0' = f'(x_0) \approx \frac{1}{h}\left(\Delta y_0 - \frac{1}{2}\Delta^2 y_0 + \frac{1}{3}\Delta^3 y_0 - \ldots + (-1)^{n-1}\frac{1}{n}\Delta^n y_0\right), \quad (3)$$

$$y_0' = f'(x_0) \approx \frac{1}{h}\left(\Delta y_{-1} + \frac{1}{2}\Delta^2 y_{-2} + \frac{1}{3}\Delta^3 y_{-3} + \ldots + \frac{1}{n}\Delta^n y_{-n}\right). \quad (4)$$

The first of them is used only for the initial rows of the table, the second for the last rows. The middle of the table usually uses the formula for numerical differentiation obtained by differentiating Stirling's interpolation polynomial:

$$y_0' = f'(x_0) \approx \frac{1}{h}\left(\frac{\Delta y_{-1} + \Delta y_0}{2} - \frac{1}{6}\frac{\Delta^3 y_{-2} + \Delta^3 y_{-1}}{2} + \right.$$
$$\left. + \frac{1}{30}\frac{\Delta^5 y_{-3} + \Delta^5 y_{-2}}{2} + \ldots\right). \quad (5)$$

This formula is more convenient for computation and has a higher accuracy as compared with the formulas (3) and (4).

Note that in using the formulas (3) to (5) we proceed from a certain correctness in the behaviour of the finite differences (see Examples 1 and 2). Each time the correctness of the difference table fails a special investigation of the behaviour of the function is required.

In some cases (for instance, when performing the computational work on a computer) it is more convenient to express derivatives not in terms of finite differences of a function, but directly in the given values of the function. Let us consider some of them which are most commonly used.

Retaining in formula (5) only the first term, we get

$$y_0' \approx \frac{\Delta y_{-1} + \Delta y_0}{2h} = \frac{y_1 - y_{-1}}{2h}. \quad (6)$$

Retaining the first two terms in the same formula, we get

$$y_0' = \frac{\Delta y_{-1} + \Delta y_0}{2h} - \frac{\Delta^3 y_{-2} + \Delta^3 y_{-1}}{12h} = \frac{y_{-2} - 8y_{-1} + 8y_1 - y_2}{12h}. \quad (7)$$

Similarly, from formula (3) we obtain

$$\left.\begin{aligned} y_0' &\approx \frac{1}{h}\left(\Delta y_0 - \frac{1}{2}\Delta^2 y_0\right) = \frac{-3y_0 + 4y_1 - y_2}{2h}, \\ y_0' &\approx \frac{1}{h}\left(\Delta y_0 - \frac{1}{2}\Delta^2 y_0 + \frac{1}{3}\Delta^3 y_0\right) = \\ &= \frac{-11y_0 + 18y_1 - 9y_2 + 2y_3}{6h}, \\ y_0' &\approx \frac{1}{h}\left(\Delta y_0 - \frac{1}{2}\Delta^2 y_0 + \frac{1}{3}\Delta^3 y_0 - \frac{1}{4}\Delta^4 y_0\right) = \\ &= \frac{-25y_0 + 48y_1 - 36y_2 + 16y_3 - 3y_4}{12h}. \end{aligned}\right\} \quad (8)$$

Approximate formulas for computing second-order derivatives are obtained by means of double differentiation of interpolation polynomials. Thus, differentiating twice Stirling's interpolation polynomial (for $q=0$), we get

$$y_0'' = f''(x_0) \approx \frac{1}{h^2}\left(\Delta^2 y_{-1} - \frac{1}{12}\Delta^4 y_{-2} + \frac{1}{90}\Delta^6 y_{-4} + \ldots\right). \quad (9)$$

For the beginning and the end of the table we use the formulas which are obtained from Newton's interpolation polynomials:

$$y_0'' = f''(x_0) \approx \frac{1}{h^2}\left(\Delta^2 y_0 - \Delta^3 y_0 + \frac{11}{12}\Delta^4 y_0 - \frac{5}{6}\Delta^5 y_0 + \ldots\right), \quad (10)$$

$$y_0'' = f''(x_0) \approx \frac{1}{h^2}\left(\Delta^2 y_{-2} + \Delta^3 y_{-3} + \frac{11}{12}\Delta^4 y_{-4} + \frac{5}{6}\Delta^5 y_{-5} + \ldots\right). \quad (11)$$

These are less convenient for computational purposes and have a lower accuracy than the formula (9).

Sometimes repeated differentiation is also carried out with the aid of formulas which express a second-order derivative directly in terms of the given values of a function. Here we confine ourselves only to one most frequently used formula of such form:

$$y_0'' \approx \frac{\Delta^2 y_{-1}}{h^2} = \frac{y_{-1} - 2y_0 + y_1}{h^2}. \quad (12)$$

Note that with an increase in the order of a derivative the accuracy of numerical differentiation is reduced abruptly. Therefore, in practical applications the formulas for numerical differentiation are seldom used with the derivatives of order higher than the second.

Example 1. The first two columns of Table 34 list the values

Table 34

	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
0.00	0.00000				
		10 017			
0.05	0.10017		100		
		10 117		101	
0.10	0.20134		201		3
		10 318		104	
0.15	0.30452		305		3
		10 623		107	
0.20	0.41075		412		
		11 035			
0.25	0.52110				

of the function $y = \sinh 2x$ with spacing $h = 0.05$. Find the values of the derivatives y' and y'' at the points $x = 0.0$ and $x = 0.1$.

Solution. Form the difference table (see Table 34) continuing it only till the differences of the fourth order, since the differences of higher orders are practically equal to zero for the chosen spacing. For numerical differentiation at the point $x=0.0$ we use formulas (3) and (10), assuming $x_0=0.0$:

$$y'|_{x=0.0} \approx \frac{1}{h}\left(\Delta y_0 - \frac{1}{2}\Delta^2 y_0 + \frac{1}{3}\Delta^3 y_0 - \frac{1}{4}\Delta^4 y_0\right) =$$
$$= 20\,(0.10017 - 0.00050 + 0.00034 - 0.00001) = 2.0000,$$

$$y''|_{x=0.0} \approx \frac{1}{h^2}\left(\Delta^2 y_0 - \Delta^3 y_0 + \frac{11}{12}\Delta^4 y_0\right) =$$
$$= 400\,(0.00100 - 0.00101 + 0.00003) = 0.008.$$

For numerical differentiation at the point $x=0.1$ use formulas (5) and (9), putting $x_0=0.1$:

$$y'|_{x=0.1} \approx \frac{1}{h}\left(\frac{\Delta y_{-1}+\Delta y_0}{2} - \frac{1}{6}\,\frac{\Delta^3 y_{-2}+\Delta^3 y_{-1}}{2}\right) =$$
$$= 20\,(0.10217 - 0.00017) = 2.0400,$$

$$y''|_{x=0.1} \approx \frac{1}{h^2}\left(\Delta^2 y_{-1} - \frac{1}{12}\Delta^4 y_{-2}\right) = 400\,(0.00201 - 0.00000) = 0.804.$$

For the sake of comparison we give the exact values of the derivatives $y'=2\cosh 2x$, $y''=4\sinh 2x$ for the points under consideration: $y'=2$, $y''=0$ for $x=0.0$; $y'=2\cosh 0.2=2.0401$, $y''=4\sinh 0.2=0.8052$ for $x=0.1$.

Example 1.2. Table 35 represents a fragment of the seven-place table of Bessel's function $y=J_0(x)$ with the spacing $h=0.02$.

Table 35

x	0.96	0.98	1.00	1.02	1.04
y	0.7825361	0.7739332	0.7651977	0.7563321	0.7473390

Compute the derivatives y' and y'' at the point $x_0=1.00$ if it is known that with such spacing (in the vicinity of the point $x=1.00$) the differences of orders higher than the third may be neglected.

Solution. Since, by hypothesis, the differences exceeding the third order may be neglected, we take advantage of formulas (7) and (12):

$$y'|_{x=1.00} \approx \frac{0.7825361 - 8\cdot 0.7739332 + 8\cdot 0.7563321 - 0.7473390}{12\cdot 0.02} =$$
$$= \frac{0.0351971 - 8\cdot 0.0176011}{0.24} = -0.440049,$$

$$y''|_{x=1.00} \approx \frac{0.7739332 - 2\cdot 0.7651977 + 0.7563321}{0.02^2} = \frac{-0.0001301}{0.0004} = -0.3252.$$

By way of comparison, here are the exact values of the derivatives $y' = -J_1(x)$, $y'' = J_1(x) - J_0(x)$ at the point under consideration: $y' = -0.4400506$, $y'' = -0.3251471$ for $x = 1.00$.

Problems

1. Given a table of the values of the function $y = f(x)$ (Table 36).

Table 36

x	y	x	y	x	y
1.0	1.2661	1.7	1.8640	2.4	3.0493
1.1	1.3262	1.8	1.9896	2.5	3.2898
1.2	1.3937	1.9	2.1277	2.6	3.5533
1.3	1.4693	2.0	2.2796	2.7	3.8417
1.4	1.5534	2.1	2.4463	2.8	4.1573
1.5	1.6467	2.2	2.6291	2.9	4.5027
1.6	1.7500	2.3	2.8296	3.0	4.8808

(a) Form the table of the values of the derivative y' at the points $x = 1.2 + 0.1 \cdot k$ $(k = 0, 1, \ldots, 16)$.

(b) Form the table of the values of the second derivative y'' at the points $x = 1.2 + 0.1 \cdot k$ $(k = 0, 1, \ldots, 16)$.

(c) With the aid of Newton's interpolation formulas find the values of the derivative y' at the points $x = 1.0$, 1.1, 2.9, 3.0.

(d) Find the values of the derivative y'' at the points $x = 1.0$, $x = 3.0$.

2. Given a table of the values of the function $y = f(x)$ (Table 37).

Table 37

x	y	x	y	x	y
0.4	0.4000	1.6	8.7826	2.8	23.3808
0.6	1.4848	1.8	10.6967	3.0	26.6819
0.8	2.6811	2.0	12.7945	3.2	30.2945
1.0	3.9983	2.2	15.0926	3.4	34.2479
1.2	5.4465	2.4	17.6093	3.6	38.5741
1.4	7.0371	2.6	20.3647	3.8	43.3084

(a) Form the table of the values of the derivative y' at the points $x = 0.8 + 0.2 \cdot k$ $(k = 0, 1, \ldots, 13)$.

(b) Form the table of the values of the second-order derivative y'' at the points $x = 0.8 + 0.2 \cdot k$ $(k = 0, 1, \ldots, 13)$.

(c) With the aid of Newton's interpolation formulas find the values of the derivative y' at the points $x=04$, 0.6, 3.6, 3.8.

(d) Find the values of the derivative y'' at the points $x=0.4$, 0.6, 3.6, 3.8.

6.2 ERRORS APPEARING IN NUMERICAL DIFFERENTIATION

Errors of two types cop up during the process of numerical differentiation of a function $y=f(x)$ given in tabular form:

(a) *Trancation errors* caused by replacing the function $f(x)$ by the interpolation polynomial $P_n(x)$;

(b) *rounding errors* due to inaccurate specification of the initial values of y_i.

Trancation errors for the formulas (3) and (4) of Sec. 6.1 are estimated by the quantity $\frac{1}{n+1}h^n|f^{(n+1)}(\xi)|$, where ξ lies in the interval (x_0, x_n) or (x_{-n}, x_0). However, this estimate is of little use, since we usually know nothing even about the order of the quantity $f^{(n+1)}(x)$. Actually, the following considerations help us to estimate the truncation error.

It is assumed that the function $f(x)$ under consideration has no rapidly varying components (i.e. components whose period does not exceed the value of the interval h). Under this condition the smallness of differences of certain order testifies to the fact that the function $f(x)$ is sufficiently well approached by a suitably chosen interpolation polynomial. If the differences of order m differ but less than by the value of the error of their rounding (see Table 38),

Table 38

The Effect of the Absolute Error ε in Values of y_i on Absolute Errors of Their Finite Differences

Difference	Δy_i	$\Delta^2 y_i$	$\Delta^3 y_i$	...	$\Delta^k y_i$
Absolute error	2ε	4ε	8ε	...	$2^k\varepsilon$

they are considered to be practically constant. Differences of higher orders are not used in the formulas for numerical differentiation, and it is considered that the truncation error does not exceed a lower-order unit of the values of y_i divided by h. But if, as it is frequently done, the formula is truncated earlier than it was indicated above, then the rejected terms serve for estimating the rounding error. For instance, if the third-order differences change suffi-

ciently smoothly, then the truncation error of formula (6) of Sec. 6.1 is approximately estimated by the quantity $\frac{1}{6h}\left|\Delta^3 y_{-2}+\frac{\Delta^3 y_{-1}}{2}\right|$; and if the fourth-order differences vary sufficiently smoothly, then the truncation error of formula (12) of Sec. 6.1 is estimated approximately by the quantity $\frac{1}{12h^2}|\Delta^4 y_{-2}|$.

Let us note that the interval of computation h enters all the formulas for estimating the truncation error in a positive power, therefore with a decrease in the interval h the truncation error, as a rule, *decreases*.

The *rounding error* is inversely proportional to the computational interval h in the formulas for the first derivative, inversely proportional to h^2 in those for the second derivative, and so on. Therefore with a decrease in h the rounding error *increases*. For estimating the rounding error the rules set forth in Chapter I are used. If, for instance, the absolute errors of the initial values of y_i do not exceed ε, then the absolute rounding errors of formulas (6), (7), (12) of Sec. 6.1 do not exceed, respectively, the quantities

$$\frac{2\varepsilon}{2h}=\frac{\varepsilon}{h}, \quad \frac{18\varepsilon}{12h}=\frac{3}{2}\frac{\varepsilon}{h}, \quad \frac{4\varepsilon}{h^2}.$$

For the formulas of type (1.8) the rounding errors are somewhat greater, being equal to

$$\frac{8\varepsilon}{2h}=4\frac{\varepsilon}{h}, \quad \frac{40\varepsilon}{6h}=6\frac{2}{3}\frac{\varepsilon}{h}, \quad \frac{128\varepsilon}{12h}=10\frac{2}{3}\frac{\varepsilon}{h},$$

respectively.

It should be underlined here that the rounding errors increase rapidly with an increase in the order of the derivative.

Example 1. Estimate the errors in computing the derivatives in Example 1 of Sec. 6.1, assuming that all the digits in the values of y_i are correct, i. e. the errors of the values of y_i are less than $\varepsilon=0.5\cdot 10^{-5}$.

Solution. In Example 1 of Sec. 6.1 use was made of the formulas for numerical differentiation which took into account all the differences up to the fourth order included. Therefore, the truncation errors are estimated by means of differences of the fifth order which do not exceed 10^{-5} (see Table 34). The value of this estimate also depends on the formula for numerical differentiation used: at the point $x=0.0$ Newton's formulas were used; at the point $x=0.1$ use was made of Stirling's formulas. The results of computations as well as the rounding errors are given in Table 39.

The above table prompts that the rounding errors considerably exceed the truncation errors, so that the total error of numerical differentiation is determined here mainly by the rounding errors. (Of course, the errors themselves are somewhat overestimated here,

Table 39

Derivative	Point	Truncation error	Rounding error
y'	$x=0.0$	$\frac{1}{5h}\lvert\Delta^5 y\rvert=4\cdot10^{-5}$	$10\frac{\varepsilon}{h}=10^{-3}$
	$x=0.1$	$\frac{1}{30h}\lvert\Delta^5 y\rvert<0.7\cdot10^{-5}$	$\frac{3}{2}\cdot\frac{\varepsilon}{h}=1.5\cdot10^{-4}$
y''	$x=0.0$	$\frac{5}{6}\frac{1}{h^2}\lvert\Delta^5 y\rvert<4\cdot10^{-3}$	$20\frac{\varepsilon}{h^2}=4\cdot10^{-2}$
	$x=0.1$	$\frac{1}{12}\frac{1}{h^2}\lvert\Delta^4 y\rvert<10^{-3}$	$4\frac{\varepsilon}{h^2}=8\cdot10^{-3}$

as is seen from the comparison with the true errors presented in Sec. 6.1.) To make the rounding errors less a *greater* value of h should have been taken.

Problems

Estimate the errors in computing the derivatives in problems 1 (a), (b), and 2 (a), (b) of Sec. 6.1, assuming that all the tabular values of y_i are given with correct digits.

6.3 CHOOSING AN OPTIMAL INTERVAL OF NUMERICAL DIFFERENTIATION

The total error in computing a derivative can be considered as a sum of the truncation error and the rounding error. Since with decrease in the interval h the truncation error reduces, and the rounding error increases, there exists an optimal interval of computation (that is to say that each formula for numerical differentiation has its own optimal interval). Thus, for formula (6) of Sec. 6.1 the truncation error does not exceed

$$\frac{h^2}{6}M_3=\frac{h^2}{6}\max_{(x_{-1},\,x_1)}|f'''(x)|,$$

and the rounding error is estimated by the quantity

$$\frac{2\varepsilon}{2h}=\frac{\varepsilon}{h},$$

where ε is the absolute error of the initial values of the function

y_i. The total error is estimated by the quantity

$$\frac{h^2}{6} M_3 + \frac{\varepsilon}{h},$$

which reaches the least value if

$$\left(\frac{h^2}{6} M_3 + \frac{\varepsilon}{h}\right)' = \frac{h}{3} M_3 - \frac{\varepsilon}{h^2} = 0,$$

i.e. if $h = \sqrt[3]{\frac{3\varepsilon}{M_3}}$, which determines the optimal interval of computation for formula (6) of Sec. 6.1. It is more convenient to write the obtained formula in the form

$$\frac{h^2}{6} M_3 = \frac{1}{2} \frac{\varepsilon}{h} \tag{1}$$

or in the form

$$\frac{1}{6h} |\Delta^3 y| \approx \frac{1}{2} \frac{\varepsilon}{h}, \tag{2}$$

which is interpreted in a simple way: the optimal interval for the formula (6) of Sec. 6.1 is determined so that the truncation error constitutes approximately half the rounding error, (the total error not exceeding $\frac{3}{2} \frac{\varepsilon}{h}$).

The optimal interval for other formulas of numerical differentiation is chosen in a similar way. For the formula (7) of Sec. 6.1 the optimal interval is determined so as to obtain the truncation error $\frac{1}{30h} |\Delta^5 y|$ approximately equal to one third the rounding error $\frac{3}{2} \frac{\varepsilon}{h}$; under this condition the total error will not exceed $2\frac{\varepsilon}{h}$. For the formula (12) of Sec. 6.1 the optimal interval is determined so that the truncation error $\frac{1}{12h^2} |\Delta^4 y|$ is approximately equal to the rounding error $4\frac{\varepsilon}{h^2}$, the total error not exceeding $8\frac{\varepsilon}{h^2}$.

In the above formulas the values $|\Delta^3 y|$, $|\Delta^4 y|$, $|\Delta^5 y|$ mean some average (for a given region of the table) absolute values of differences of an appropriate order.

Example 1. Let us follow the alteration of the error in computing a derivative with the decreasing interval h, taking the function $y = e^x$ for which $y' = e^x = y$.

Solution. Take the four-digit table of values of the function e^x, which means the absolute error $\varepsilon = 0.5 \cdot 10^{-4}$. Use the simplest formula of numerical differentiation (6) of Sec. 6.1 with the interval h reducing from 0.10 to 0.01, putting $x_0 = 1.15$. The compu-

Table 40

Numerical Differentiation with the Interval h

$h = 0.10$

x	$y = e^x$	$D_h y_0$	Error
1.05	2.8577		
1.15	3.1582	3.1630	—0.0048
1.25	3.4903		

$h = 0.09$

x	$y = e^x$	$D_h y_0$	Error
1.06	2.8864		
1.15	3.1582	3.1622	—0.0040
1.24	3.4556		

$h = 0.08$

x	$y = e^x$	$D_h y_0$	Error
1.07	2.9154		
1.15	3.1582	3.1613	—0.0031
1.23	3.4212		

$h = 0.07$

x	$y = e^x$	$D_h y_0$	Error
1.08	2.9447		
1.15	3.1582	3.1607	—0.0025
1.22	3.3872		

$h = 0.06$

x	$y = e^x$	$D_h y_0$	Error
1.09	2.9743		
1.15	3.1582	3.1600	—0.0018
1.21	3.3535		

$h = 0.05$

x	$y = e^x$	$D_h y_0$	Error
1.10	3.0042		
1.15	3.1582	3.1590	—0.0008
1.20	3.3201		

$h = 0.04$

x	$y = e^x$	$D_h y_0$	Error
1.11	3.0344		
1.15	3.1582	3.1588	—0.0006
1.19	3.2871		

$h = 0.03$

x	$y = e^x$	$D_h y_0$	Error
1.12	3.0649		
1.15	3.1582	3.1583	—0.0001
1.18	3.2544		

$h = 0.02$

x	$y = e^x$	$D_h y_0$	Error
1.13	3.0957		
1.15	3.1582	3.1575	0.0007
1.17	3.2220		

$h = 0.01$

x	$y = e^x$	$D_h y_0$	Error
1.14	3.1268		
1.15	3.1582	3.1550	0.0032
1.16	3.1899		

tations are presented in Table 40, in which the following notation is introduced: $\frac{y_1 - y_{-1}}{2h} = D_h y_0$ (for the sake of brevity).

As is seen from the table, with the interval h decreasing from 0.10 to 0.03 the absolute value of the error of numerical differentiation suffers a decrease (from 0.0048 to 0.0001), followed by another increase (up to 0.0032 for $h = 0.01$) with a further decrease in the interval h. This means that for the interval $h \leqslant 0.03$ the main role is played by the rounding errors.

Here the optimal interval of numerical differentiation is readily computed by formula (1):

$$h_{\text{opt}} = \sqrt[3]{\frac{3\varepsilon}{M_3}} = \sqrt[3]{\frac{0.00015}{3.5}} \approx 0.035.$$

Table 41 presents the results of numerical differentiation with this interval.

Table 41

Numerical Differentiation with the Interval $h = 0.035$

x	$y = e^x$	$D_h y_0$	Error
1.115 1.15 1.185	3.0496 3.1582 3.2707	3.1586	—0.0004

The comparison of Table 40 and 41 shows that the optimal interval h is determined by the formula (1) in a very rough way; in the example under consideration the least error is obtained with $h = 0.03$.

Example 2. Choose a suitable formula and optimal interval of numerical differentiation for the function presented by Table 42. The values of the function y are given with all correct digits.

Solution. Here the absolute error of the initial values of the function does not exceed $\varepsilon = 0.5 \cdot 10^{-4}$. In numerical differentiation with the aid of formula (6) of Sec. 6.1 the optimal interval is determined by the condition

$$\frac{1}{6}\Delta^3 y \approx \frac{1}{2}\varepsilon.$$

Since in Table 42 the finite differences of the third order vary from $-3 \cdot 10^{-4}$ to $3 \cdot 10^{-4}$, the quantity $\frac{1}{6}\Delta^3 y$ changes from $-0.5 \cdot 10^{-4}$ to $0.5 \cdot 10^{-4}$, i. e., on an average, has a suitable order. Therefore it is possible to compute derivatives using formula (6) of Sec. 6.1 with the interval $h = 0.1$. This gives a total error of the order

Table 42

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$
1.0	0.4401			
		308		
1.1	0.4709		—34	
		274		—3
1.2	0.4983		—37	
		237		—1
1.3	0.5220		—38	
		199		—1
1.4	0.5419		—39	
		160		—1
1.5	0.5579		—40	
		120		—1
1.6	0.5699		—41	
		79		—1
1.7	0.5778		—42	
		37		2
1.8	0.5815		—40	
		—3		—2
1.9	0.5812		—42	
		—45		3
2.0	0.5767		—39	
		—84		0
2.1	0.5683		—39	
		—123		1
2.2	0.5560		—38	
		—161		2
2.3	0.5399		—36	
		—197		2
2.4	0.5202		—34	
		—231		2
2.5	0.4971		—32	
		—263		3
2.6	0.4708		—29	
		—292		2
2.7	0.4416		—27	
		—319		
2.8	0.4097			

$1.5\frac{\varepsilon}{h}$ to $2\frac{\varepsilon}{h}$, i. e. of the order 10^{-3}. To reduce the total error let us try to increase the interval h and make use of a more accurate formula (7) of Sec. 6.1. Doubling the interval h yields the data tabulated below.

As is seen from Table 43, for the interval $h_1 = 2h = 0.2$ the average absolute values of the fifth-order differences are equal to $3.6 \cdot 10^{-4}$ so that $\frac{1}{30}|\Delta^5 y| \approx 1.2 \cdot 10^{-5}$, which constitutes roughly a quarter from $\varepsilon = 0.5 \cdot 10^{-4} = 5 \cdot 10^{-5}$. If we choose such an interval,

Table 43

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$	$\Delta^5 y$
1.0	0.4401					
		582				
1.2	0.4983		—146			
		436		—10		
1.4	0.5419		—156		2	
		280		—8		6
1.6	0.5699		—164		8	
		116		0		—3
1.8	0.5815		—164		5	
		—48		5		—2
2.0	0.5767		—159		3	
		—207		8		4
2.2	0.5560		—151		7	
		—358		15		—3
2.4	0.5202		—136		4	
		—494		19		
2.6	0.4708		—117			
		—611				
2.8	0.4097					

then the truncation error will be equal to $\frac{1}{30h_1}|\Delta^5 y| \approx 0.6 \cdot 10^{-4}$, and the rounding error to $\frac{3}{2}\frac{\varepsilon}{h} \approx 4 \cdot 10^{-4}$; hence, the total error is estimated by the quantity $5 \cdot 10^{-4}$. This is two or three times better than with the interval $h = 0.1$, but there still remains a possibility for a further increase of the interval.

With an increase in the interval of computation by λ times the first-order differences are increased λ times, the second-order differences λ^2 times, ..., the fifth-order differences λ^5 times. Therefore the interval $h_2 = \frac{3}{2}h_1 = 3h = 0.3$ gives the truncation error approximately equal to $\frac{1}{30h_2} \cdot (1.5)^5 \cdot 4 \cdot 10^{-4} = 3 \cdot 3 \cdot 10^{-4}$, the rounding error being equal to $\frac{3}{2}\frac{\varepsilon}{h_2} = 2.5 \cdot 10^{-4}$, which is quite acceptable. A further increase of the interval is hardly advisable, especially so because Table 43 lacks the data necessary to compile a table of differences up to the fifth order with the interval $h = 0.4$. Thus, $h_2 = 0.3$ is chosen as the optimal interval. This interval is suitable for computing the values of derivatives only at the points $x = 1.6$; 1.7; ...; 2.2. In this case it is convenient to use the formula (7) of Sec. 6.1 in the *non-difference* form exp-

ression, i.e. in the form

$$y_0' \approx \frac{1}{12h}(y_{-2} - 8y_{-1} + 8y_1 - y_2),$$

where y_i should be understood as $y(x_0 + ih_2)$. For instance,

$$y'|_{x=1.6} \approx \frac{1}{12 \cdot 0.3}(0.4401 - 8 \cdot 0.5220 + 8 \cdot 0.5812 - 0.5560) = 0.0994,$$

$$y'|_{x=1.7} \approx \frac{1}{12 \cdot 0.3}(0.4709 - 8 \cdot 0.5419 + 8 \cdot 0.5767 - 0.5399) = 0.0582.$$

At the points $x = 1.4$; 1.5 and $x = 2.3$; 2.4 we can also make use of formula (7) of Sec. 6.1 with the interval $h_1 = 0.2$ which is rather close to the optimal one and ensures almost the same accuracy. At points situated still closer to the boundaries of the table we have to use formula (6) of Sec. 6.1 with the interval $h = 0.1$ or even Newton's formulas.

In conclusion it should be underlined once again that the mentioned estimates usually turn out to be somewhat overestimated. Thus, in the considered example the true errors in computing the derivatives are several times smaller than the mentioned estimates. For the sake of comparison, let us compute the value of the derivative y' at the point $x = 1.6$ using formula (6) of Sec. 6.1 with the interval $h = 0.1$. We get

$$y'|_{x=1.6} \approx \frac{1}{2 \cdot 0.1}(0.5778 - 0.5579) = 0.0995,$$

which differs from the accurate value 0.0992 only by $3 \cdot 10^{-4}$.

Problems

Choose the optimal interval of numerical differentiation and a suitable formula in Problems 1 (a), (b), and 2 (a), (b) of Sec. 6.1, assuming that all the tabular values of y_i are given with correct digits. In Problems 1 (6) and 2 (b) find y'' at the point $x = 1.6$.

7

APPROXIMATE COMPUTATION OF INTEGRALS

7.1 QUADRATURE FORMULAS WITH EQUALLY SPACED POINTS

Replacing the integrand by some interpolation polynomial, we get the *quadrature formulas* of the form

$$\int_a^b f(x)\,dx = \sum_{k=0}^{n} A_k f(x_k) + R, \tag{1}$$

where x_k are the chosen interpolation points; A_k, the coefficients depending only on the choice of the points, but not on the form of the function ($k=0, 1, \ldots, n$); R, the remainder term, or the error of the quadrature formula. Discarding the remainder error R, we introduce the *truncation error*. In the process of computation various rounding errors are added to it.

Subdivide the interval of integration $[a, b]$ into n equal parts by the system of points

$$x_i = x_0 + ih \quad (i=0, 1, \ldots, n), \quad x_0 = a, \quad x_n = b, \quad h = \frac{b-a}{n},$$

and compute the integrand at the obtained points

$$y_i = f(x_i) \quad (i=0, 1, \ldots, n).$$

The quadrature formulas for equally spaced points are called the *Newton-Cotes formulas* (see [1], [12], [21]). The Newton-Cotes formulas differ in the degree of the interpolation polynomial used. To get rid of high-degree polynomials it is standard practice to subdivide the interval of integration into separate subintervals, to use the Newton-Cotes formulas of lower orders on each subinterval and then to add the obtained results (which yields so-called *compound formulas*). Some simple formulas of such type are considered below.

1. The *trapezoidal formula*:

$$\int_a^b f(x)\,dx \approx h\left(\frac{y_0+y_n}{2}+y_1+y_2+\ldots+y_{n-1}\right), \qquad (2)$$

where $y_i=f(x_i)$ $(i=0, 1, \ldots, n)$.

The remainder term has the form

$$R_1=-\frac{nh^3}{12}f''(\xi)=-\frac{(b-a)h^2}{12}f''(\xi), \quad a<\xi<b.$$

The trapezoidal formula yields the exact value of the integral when the integrand $f(x)$ is a linear function, since we have then $f''(x)\equiv 0$.

2. *Simpson's formula (parabolas formula)*:

$$\int_a^b f(x)\,dx \approx \frac{h}{3}[y_0+y_{2m}+2(y_2+y_4+\ldots+y_{2m-2})+$$
$$+4(y_1+y_3+\ldots+y_{2m-1})], \qquad (3)$$

where

$$h=\frac{b-a}{n}=\frac{b-a}{2m}.$$

The remainder term has the form

$$R_2=-\frac{mh^5}{90}f^{(4)}(\xi)=-\frac{(b-a)h^4}{180}f^{(4)}(\xi), \quad a<\xi<b.$$

Simpson's formula is exact for polynomials up to degree three, since in this case $f^{(4)}(x)=0$.

Note that in Simpson's formula the number of points is obligatory odd, i.e. n is even, $n=2m$.

3. *Newton's formula (three-eighths rule)*:

$$\int_a^b f(x)\,dx \approx \frac{3h}{8}[y_0+y_{3m}+2(y_3+y_6+\ldots+y_{3m-3})+$$
$$+3(y_1+y_2+y_4+y_5+\ldots+y_{3m-2}+y_{3m-1})], \qquad (4)$$

where

$$h=\frac{b-a}{n}=\frac{b-a}{3m}.$$

The remainder term has the form

$$R_3=-\frac{3mh^5}{80}f^{(4)}(\xi)=-\frac{(b-a)h^4}{80}f^{(4)}(\xi), \quad a<\xi<b.$$

Note that in formula (4) the number of points is obligatory equal to $3m+1$, i.e. $n=3m$.

If the function $y=f(x)$ is given in a tabular form and it is difficult to find its derivatives, then in assumption of absence of rapidly fluctuating components we may apply approximate formulas for computing errors expressed in terms of finite differences:

$$R_1 \approx -\frac{b-a}{12}\overline{\Delta^2 y}, \tag{5}$$

$$R_2 \approx -\frac{b-a}{180}\overline{\Delta^4 y}, \tag{6}$$

$$R_3 \approx -\frac{b-a}{80}\overline{\Delta^4 y}, \tag{7}$$

where $\overline{\Delta^2 y}$, $\overline{\Delta^4 y}$ denote the arithmetic mean value of the differences of the appropriate order.

Example 1. Evaluate the integral

$$\int_0^1 e^{-x^2}dx,$$

employing the trapezoidal rule for $n=10$ and estimate the error of computation.

Solution. First estimate the remainder term. To this end find the second derivative of the function $y=e^{-x^2}$:

$$y''=2(2x^2-1)e^{-x^2}.$$

On the interval [0, 1] the modulus of the second derivative $|y''(x)|$ attains the maximum value at $x=0$. Thus, we have

$$|R_1| \leqslant \frac{\max|y''(x)|}{12}|b-a|h^2 = \frac{2\cdot(0.1)^2}{12} < 0.002.$$

To remove the effect of the rounding error on the accuracy of the result we shall carry out the computations with one extra digit, i.e. with four decimal digits. Form a table of values of the integrand function (Table 44).

By the trapezoidal formula we get

$$\int_0^1 e^{-x^2}dx \approx 0.1\cdot 7.4620 = 0.7462.$$

The final answer is rounded to three digits:

$$\int_0^1 e^{-x^2}dx \approx 0.746.$$

Example 2. Compute the integral

$$\int_0^1 e^{x^2}dx$$

Table 44

Values of the Function $y=e^{-x^2}$

i	x_i	x_i^2	y_i	i	x_i	x_i^2	y_i
0	0	0	1.0000	6	0.6	0.36	0.6977
1	0.1	0.01	0.9900	7	0.7	0.49	0.6126
2	0.2	0.04	0.9608	8	0.8	0.64	0.5273
3	0.3	0.09	0.9139	9	0.9	0.81	0.4449
4	0.4	0.16	0.8521	10	1.0	1.00	0.3679
5	0.5	0.25	0.7788				

$$\frac{1}{2}(y_0+y_{10})+\sum_{i=1}^{9} y_i=7.4620$$

by Simpson's formula for $n=10$ and estimate the remainder term.

Solution. Let us estimate the remainder term by finding the fourth derivative of the function $y=e^{x^2}$:

$$y^{(4)}(x)=4(4x^4+12x^2+3)e^{x^2}.$$

The derivative $y^{(4)}(x)$ attains the greatest value on the interval $[0, 1]$ at $x=1$. Thus,

$$|R_2|\leqslant\frac{5\cdot(0.1)^5}{90}\cdot 76\cdot 2.718\approx 0.000115.$$

Form a table of values of the integrand function $y=e^{x^2}$ (Table 45), entering the ordinates with even and odd numbers in different columns. The last row of the table presents the results of columnwise summation. Then, applying Simpson's formula, we find

$$\int_0^1 e^{x^2}dx\approx\frac{1}{30}(3.7183+4\cdot 7.2685+2\cdot 5.5441)=1.46268.$$

The final answer is rounded to four digits:

$$\int_0^1 e^{x^2}dx\approx 1.4627.$$

Example 3. Compute the integral

$$\int_0^{0.6}\frac{dx}{1+x}$$

Table 45

Values of the Function $y=e^{x^2}$

i	x_i	x_i^2	Values of $y_i=e^{x_i^2}$		
			for $i=0$, $i=10$	for even i	for odd i
0	0.0	0.00	1.0000		
1	0.1	0.01			1.0101
2	0.2	0.04		1.0408	
3	0.3	0.09			1.0942
4	0.4	0.16		1.1735	
5	0.5	0.25			1.2840
6	0.6	0.36		1.4333	
7	0.7	0.49			1.6323
8	0.8	0.64		1.8965	
9	0.9	0.81			1.2479
10	1.0	1.00	2.7189		
		Σ	3.7188	5.4441	7.2685

using Newton's formula (the three-eighths rule) for $h=0.1$ and estimate the remainder term.

Solution. To estimate the remainder term find the fourth derivative of the integrand function $y=(1+x)^{-1}$:

$$y^{(4)}=24\,(1+x)^{-5}.$$

On the interval [0; 0.6] it has the greatest value at $x=0$. Therefore

$$|R_3|\leqslant\frac{6\cdot(0.1)^5}{80}\cdot 24=1.8\cdot 10^{-5}.$$

Form a table of values of the integrand function (Table 46), entering these values in the three columns depending on the number.

Using the results of computations, by formula (4) we find

$$\int_0^{0.6}\frac{dx}{1+x}\approx\frac{3}{8}0.1\cdot(1.62500+1.53846+9.37014)=0.47001.$$

Example 4. Evaluate the integral $\int_0^{0.8} f(x)\,dx$ by Simpson's formula and estimate the remainder term if the integrand function

Table 46

Values of the Function $y=\frac{1}{1+x}$

i	x_i	$1+x_i$	Values of $y_i=\frac{1}{1+x_i}$		
			for $i=0$, $i=6$	for $i=3$	for $i=1, 2, 4, 5$
0	0.0	1.0	1.00000		
1	0.1	1.3			0.90909
2	0.2	1.2			0.83333
3	0.3	1.3		0.76923	
4	0.4	1.4			0.71429
5	0.5	1.5			0.66667
6	0.6	1.6	0.62500		
		Σ	1.62500	0.76923	3.12338

is specified in a tabular form

x	0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8
$f(x)$	1.0000	0.9950	0.9801	0.9553	0.9211	0.8776	0.8253	0.7648	0.6967

Solution. Put $h=0.1$ (the spacing of the table), then $n=2m=8$. The sums necessary for computation are calculated in Table 47.

Employing the results obtained, by formula (3) we find

$$\int_0^{0.8} f(x)\,dx \approx \frac{1}{3}(1.6967+4\cdot 3.5927=2\cdot 2.7265)=0.71735.$$

To estimate the errors form a table of finite differences for the given function (Table 48) and take advantage of formula (6). The average value of the fourth-order differences in Table 48 is equal to $\overline{\Delta^4 y}\approx 2\cdot 10^{-4}$; therefore by formula (6) we get

$$R_2 \approx \frac{0.8\cdot 2\cdot 10^{-4}}{180}\approx 10^{-6},$$

thus, all digits in the obtained result may be considered to be correct.

Table 47

Values of the Function $f(x)$

i	x_i	Values of $f(x_i)$		
		for $i=0$ and $i=8$	for odd i	for even i
0	0	1.0000		
1	0.1		0.9950	
2	0.2			0.9801
3	0.3		0.9553	
4	0.4			0.9211
5	0.5		0.8776	
6	0.6			0.8253
7	0.7		0.7648	
8	0.8	0.6967		
Σ		1.6967	3.5927	2.7265

Table 48

Finite Differences for the Function $y=f(x)$

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
0	1.0000	−50	−99	0	+5
0.1	0.9950	−149	−99	5	−4
0.2	0.9801	−248	−94	1	+4
0.3	0.9553	−342	−93	5	+1
0.4	0.9211	−435	−88	6	0
0.5	0.8776	−523	−82	6	
0.6	0.8253	−605	−76		
0.7	0.7648	−681			
0.8	0.6967				

Problems

1. Evaluate approximately, by the trapezoidal formula, the integral

$$\int_0^1 (3x^2-4x)\,dx,$$

putting $n=10$. Compute the exact value of this integral and find the absolute and the relative errors of the result.

2. Evaluate the integral

$$\int_0^1 \frac{x\,dx}{1+x}$$

using Simpson's formula, putting $n=10$. Estimate the absolute error Δ of the result, making use of the formula for the remainder term.

In the following problems, evaluate approximately the integrals by the indicated formulas and estimate the remainder term R.

3. $\int_1^5 \frac{dx}{x}$ by the trapezoidal formula for $n=4$.

4. $\int_1^9 \sqrt{6x-5}\,dx$ by the trapezoidal formula for $n=8$.

5. $\int_0^{1.2} \ln(1+x^2)\,dx$

(a) by the trapezoidal formula for $n=6$, (b) by Simpson's formula for $n=6$.

6. $\int_0^1 \sin x^2\,dx$

(a) by the trapezoidal formula for $n=10$, (b) by Simpson's formula for $n=10$.

7. $\int_0^1 \cos x^2\,dx$

(a) by the trapezoidal formula for $n=10$, (b) by Simpson's formula for $n=10$.

8. $\int_4^{5.2} \ln x\,dx$

(a) by the trapezoidal formula for $n=6$, (b) by Simpson's formula for $n=6$.

9. $\int_1^3 \frac{dx}{1+x}$

(a) by the trapezoidal formula for $n=4$, (b) by Simpson's formula for $n=4$.

10. $\int_{0.1}^{1.6} \frac{dx}{\sqrt{x}}$ by the three-eighths rule for $n=30$.

7.2 CHOOSING THE INTERVAL OF INTEGRATION

The problem consists in choosing the interval h to ensure the specified accuracy ε of evaluating the integral by a chosen formula of numerical integration.

Let us consider two methods of solving this problem.

1. Choosing the interval by estimating the remainder term. Let it be required to evaluate an integral accurate to ε. Using the formula for the corresponding remainder term R, choose h such that the inequality $|R| < \frac{\varepsilon}{2}$ is fulfilled. Then evaluate the integral by the approximate formula with the interval obtained, the number of digits being such that the rounding error does not exceed $\frac{\varepsilon}{2}$.

Note. There are cases when the permissible error ε is devided between the truncation error and the rounding error not in equal shares. For instance, if the computation of values of the integrand function turns out to be a labour-consuming operation but can be carried out with any preassigned accuracy, then it might be expedient to choose h from the condition $|R| < \varepsilon$. Another extreme case may happen for the functions specified experimentally when it is difficult to ensure a high accuracy of computation of values of a function.

Example 1. With the aid of Simpson's formula evaluate the integral

$$\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \frac{\sin x}{x}\, dx$$

accurate to $\varepsilon = 10^{-3}$.

Solution. First choose the interval of integration.

The remainder term of Simpson's formula has the form

$$R = -\frac{h^4 (b-a)}{180} f^{(4)}(\xi), \quad a < \xi < b.$$

Choose the interval h so that the following inequality is satisfied

$$\frac{h^4 (b-a)}{180} \max_{[a,\, b]} |f^{(4)}(x)| < 0.5 \cdot 10^{-3}.$$

Compute $f^{(4)}(x)$:

$$f^{(4)}(x) = \frac{\sin x}{x} + 4\frac{\cos x}{x^2} - 12\frac{\sin x}{x^3} - 24\frac{\cos x}{x^4} + 24\frac{\sin x}{x^5}.$$

In estimating $|f^{(4)}(x)|$ on the interval $[\pi/4, \pi/2]$ take advantage of the fact that the quantities

$$\frac{\sin x}{x}\left(1 - \frac{12}{x^2} + \frac{24}{x^4}\right) \quad \text{and} \quad 4\frac{\cos x}{x^2}\left(\frac{6}{x^2} - 1\right)$$

are positive and decrease on this interval. Therefore they reach the greatest value at the point $x=\frac{\pi}{4}$, and

$$|f^{(4)}(x)| \leqslant \frac{\sin x}{x}\left(1-\frac{12}{x^2}+\frac{24}{x^4}\right)+4\frac{\cos x}{x^2}\left(\frac{6}{x^2}-1\right)<81.$$

Thus, for determining the interval of computation h we obtain the inequality

$$\frac{h^4\cdot\frac{\pi}{4}}{180}\cdot 81<0.5\cdot 10^{-3},$$

whence $h^4<14\cdot 10^{-4}$ and $h<1.9\cdot 10^{-1}=0.19$.

On the other hand, the spacing h should be chosen so as to subdivide the interval $\left[\frac{\pi}{4}, \frac{\pi}{2}\right]$ into an even number of equal parts. The two mentioned conditions are met by the following value: $h=\frac{\pi}{24}=0.13<0.19$, for which $n=\frac{b-a}{h}=6$. To ensure the error not exceeding $0.5\cdot 10^{-3}$ it is sufficient to carry out the computations accurate to four decimal places.

Form a table of values of the function $y=\frac{\sin x}{x}$ with $h=\frac{\pi}{24}=7°30'=0.13090$ (Table 49), the last row containing the results of summation for the last three columns.

Then, using Simpson's formula, we find for $n=6$

$$\int_{\pi/4}^{\pi/2}\frac{\sin x}{x}dx\approx\frac{h}{3}[(y_0+y_6)+4(y_1+y_3+y_5)+2(y_2+y_4)]=$$

$$=0.043633(1.5369+4\cdot 2.3386+2\cdot 1.5649)=0.6118.$$

The final result is rounded to three digits:

$$\int_{\pi/4}^{\pi/2}\frac{\sin x}{x}dx\approx 0.612.$$

2. Double computation. Since the finding of $\max|f^{(k)}(x)|$ often involves many cumbersome calculations, the following method is used in computational practice: integral I is evaluated by the chosen quadrature formula twice—first for a spacing h and then for $\frac{h}{2}$, i.e. with the number n doubled.

Denoting the results of computation by I_n and I_{2n}, respectively, compare them. If $|I_n-I_{2n}|<\varepsilon$, where ε is the permissible error, then put

$$I\approx I_{2n}.$$

Table 49

Values of the Function $y=\frac{\sin x}{x}$

i	x_i^0	x_i	$\sin x$	y_0, y_6	y_{2m}	y_{2m-1}
0	45°00′	0.7854	0.7071	0.9003		
1	52°30′	0.9163	0.7934			0.8659
2	60°00′	1.0472	0.8660		0.8270	
3	67°30′	1.1781	0.9239			0.7842
4	75°00′	1.3090	0.9659		0.7379	
5	82°30′	1.4399	0.9914			0.6885
6	90°00′	1.5708	1.0000	0.6366		
			Σ	1.5369	1.5649	2.3386

But if it turns out that $|I_n-I_{2n}|\geqslant\varepsilon$, then the computation is repeated taking the spacing $\frac{h}{4}$. Sometimes a number close to $\sqrt[m]{\varepsilon}$ may be recommended as the initial spacing, where $m=2$ for the trapezoidal formula and $m=4$ for Simpson's formula.

This method is widely used in evaluating integrals on computers, since it inables us to realize an automatic choice of the spacing for the preassigned accuracy with a simultaneous check of the computations.

Note that for an approximate estimate of the truncation error Δ we can use *Runge's principle*, according to which

$$\Delta\approx\frac{1}{3}|I_n-I_{2n}| \text{ for the trapezoidal formula,}$$

$$\Delta\approx\frac{1}{15}|I_n-I_{2n}| \text{ for Simpson's formula.}$$

Example 2. Compute the following integral by Simpson's for-

mula

$$\int_0^{\pi} \frac{dx}{x+\cos x}$$

accurate to $\varepsilon = 3 \cdot 10^{-4}$.

Solution. Subdivide the interval $[0, \pi]$ into eight parts and find the values of the integrand function at the points of division (Table 50). Then evaluate the integral by Simpson's formula first for the spacing $h = \frac{\pi}{8}$ $(n = 8)$, and then for $h = \frac{\pi}{4}$ $(n = 4)$.

The results of computations are tabulated below (Table 50). Underlined in the column y_i of this table are the values of y_i used in evaluating the integral with the spacing $h = \frac{\pi}{4}$. The columns m_i' and m_i indicate the coefficients used with the ordinates y_i in Simpson's formula when carrying out computations with the spacing $h = \frac{\pi}{8}$ and $h = \frac{\pi}{4}$, respectively. The last row of the table presents the respective sums $\sum m_i' y_i$ and $\sum m_i y_i$.

By Simpson's formula we find: for $n = 4$, $h = \pi/4 = 0.78540$, $h/3 = 0.26180$

$$I_4 = 0.26180 \cdot 7.8457 = 2.0540;$$

for $n = 8$, $h = \pi/8 = 0.39270$, $h/3 = 0.13090$

$$I_8 = 0.13090 \cdot 15.6161 = 2.0441.$$

Table 50

Values of the Function $y = \frac{1}{x+\cos x}$

i	x_i^0	x_i	$\cos x_i$	$x_i + \cos x_i$	y_i	m_i'	m_i
0	0°	0	1.0000	1.0000	1.0000	1	1
1	22°30′	0.3927	0.9239	1.3166	0.7595	4	
2	45°	0.7854	0.7071	1.4925	0.6700	2	4
3	67°30′	1.1781	0.3827	1.5608	0.6407	4	
4	90°	1.5708	0.0000	1.5708	0.6366	2	2
5	112°30′	1.9635	−0.3827	1.5808	0.6326	4	
6	135°	2.3562	−0.7071	1.6491	0.6064	2	4
7	157°30′	2.7489	−0.9239	1.8250	0.5480	4	
8	180°	3.1416	−1.0000	2.1416	0.4669	1	1
					$\sum$	15.6161	7.8457

Comparing the values I_4 and I_8, we get

$$|I_4 - I_8| = 0.0099 > 3 \cdot 10^{-4}.$$

Moreover, even $\frac{1}{15}|I_4 - I_8| > 3 \cdot 10^{-4}$, therefore the error of the more accurate value I_8 is still not sufficient.

Consequently the spacing should be reduced. Carry out analogous computations with the spacings $h = \frac{\pi}{8}$ and $h = \frac{\pi}{16}$. The results of computations for $h = \frac{\pi}{8}$ are entered in table 50, those for $h = \frac{\pi}{16}$ in Table 51. For $h = \frac{\pi}{16} = 0.19635$ we get $I_{16} = 0.06545 \cdot 31.2169 = 2.0431$.

The comparison of I_8 and I_{16} gives

$$|I_8 - I_{16}| = 0.0010.$$

According to Runge's rule, the error of the more accurate value I_{16} does not exceed $\frac{1}{15} \cdot 0.0010 < 3 \cdot 10^{-4}$. Therefore, the value I_{16}

Table 51

Values of the Function $y = \frac{1}{x + \cos x}$ *with the Interval* $h = \frac{\pi}{16}$

i	x_i^0	x_i	$\cos x_i$	$x_i + \cos x_i$	y_i	m_i
0	0°	0	1.0000	1.0000	1.0000	1
1	11°15′	0.1963	0.9808	1.1771	0.8495	4
2	22°30′	0.3927	0.9239	1.3166	0.7595	2
3	33°45′	0.5890	0.8315	1.4205	0.7040	4
4	45°	0.7854	0.7071	1.4925	0.6700	2
5	56°15′	0.9817	0.5554	1.5371	0.6506	4
6	67°30′	1.1781	0.3827	1.5608	0.6407	2
7	78°45′	1.3744	0.1951	1.5695	0.6371	4
8	90°	1.5708	0.	1.5708	0.6366	2
9	101°15′	1.7670	—0.1951	1.5719	0.6361	4
10	112°30′	1.9635	—0.3827	1.5808	0.6326	2
11	123°45′	2.1599	—0.5554	1.6045	0.6232	4
12	135°	2.3562	—0.7071	1.6491	0.6064	2
13	146°15′	2.5525	—0.8315	1.7210	0.5810	4
14	157°30′	2.7489	—0.9239	1.8250	0.5480	2
15	168°45′	2.9452	—0.9808	1.9644	0.5090	4
16	180°	3.1416	—1.0000	2.1416	0.4669	1
					Σ	31.2169

is computed to the required accuracy. Thus

$$\int_0^{\pi} \frac{dx}{x+\cos x} = 2.0432.$$

Problems

In Problems 1 to 3 evaluate the integrals by the trapezoidal formula to the preassigned accuracy, determining h by estimating the remainder term.

1. $\int_0^1 \frac{dx}{1+x}$ to within 10^{-2}.

2. $\int_0^1 \frac{dx}{1+x^2}$ to within 10^{-2}.

3. $\int_0^1 \frac{dx}{\sqrt{1+x^2}}$ to within 10^{-5}.

In Problems 4 to 33 evaluate the integrals to the indicated accuracy. Determine h ensuring the required accuracy with the aid of double computation.

Evaluate the following integrals by the trapezoidal formula to within 10^{-2}.

4. $\int_0^1 \frac{dx}{1+x^3}$. 5. $\int_1^2 x \log x\, dx$. 6. $\int_1^2 \frac{\log x}{x} dx$.

7. $\int_1^2 \frac{\cos x}{x} dx$. 8. $\int_0^{\pi/2} \frac{\cos x}{1+x} dx$.

Compute the following integrals by Simpson's formula accurate to 10^{-4}:

9. $\int_0^{0.2} \cos \frac{\pi x^2}{2} dx$. 10. $\int_0^1 \frac{\ln(1+x)}{1+x^2} dx$.

11. $\int_0^1 \frac{\arctan x}{x} dx$. 12. $\int_0^{0.5} \frac{dx}{\sqrt{(1-x^2)(1-0.25x^2)}}$.

13. $\int_0^{0.5} \frac{dx}{\sqrt{(1-x^2)(1-0.75x^2)}}$. 14. $\int_0^{0.5} \cos \frac{x^2}{4} dx$.

15. $\int_0^1 \frac{\ln(1+x^2)}{1+x^2}\,dx$. 16. $\int_0^{0.5} \frac{(\arctan x)^2}{x}\,dx$.

17. $\int_0^{\pi/3} \frac{dx}{\sqrt{1-0.25\sin^2 x}}$. 18. $\int_0^{0.5} \sqrt{\frac{1-0.25x^2}{1-x^2}}\,dx$.

19. $\int_0^{0.5} \sqrt{\frac{1-0.75x^2}{1-x^2}}\,dx$.

Compute the following integrals by Simpson's formula accurate to 10^{-3}:

20. $\int_0^1 \sqrt{x}\sin x\,dx$. 21. $\int_0^1 \sqrt{x}\cos x\,dx$. 22. $\int_0^{\pi} \frac{dx}{1+x+\sqrt{\sin x}}$.

23. $\int_0^{\pi/2} \frac{dx}{x+\sqrt{\cos x}}$. 24. $\int_2^3 \frac{dx}{1+\sqrt{\ln x}}$. 25. $\int_0^{\pi} \frac{dx}{1+\sin^3 x}$.

26. $\int_0^{\pi/2} \frac{dx}{1+\cos^2 x}$. 27. $\int_0^1 e^{-5x^3+x+0.5}\,dx$. 28. $\int_0^1 e^{-4x^3+2x+1}\,dx$.

29. $\int_0^1 \frac{\sin x}{x^2+1}\,dx$. 30. $\int_0^1 \frac{\ln(1+\sqrt{x})}{\sqrt[3]{x}}\,dx$.

31. $\int_0^{\pi/2} \sqrt{1-0.5\sin^2 x}\,dx$.

Compute the following integral by Simpson's formula accurate to 10^{-4}:

32. $I_1=\int_0^{\pi/2} \frac{\sin ax}{x}\,dx$, $I_2=\int_{0.5}^{1.5} \frac{e^{ax}}{x}\,dx$, $a=0{,}05k$, $k=2, 3, \ldots, 22$.

33. $I_1=\int_0^1 \frac{e^{-x^2}\sin ax}{a+x^2}\,dx$, $I_2=\int_0^1 \frac{\sqrt{a+x^2}}{1+\cos ax}\,dx$, $a=0.5+0.1k$, $k=0, 1, 2, \ldots, 15$.

34. Find the approximate value of the integral $\int_0^2 f(x)\,dx$ by Simpson's formula with double computation if the integrand function $f(x)$ is presented by Table 52.

Find the error R using Runge's rule.

Table 52

x	$f(x)$	x	$f(x)$	x	$f(x)$
0.0	1.00000	0.7	0.61263	1.4	0.14086
0.1	0.99005	0.8	0.52729	1.5	0.10540
0.2	0.96079	0.9	0.44486	1.6	0.07730
0.3	0.91393	1.0	0.36788	1.7	0.05558
0.4	0.85214	1.1	0.29820	1.8	0.03916
0.5	0.77880	1.2	0.23693	1.9	0.02705
0.6	0.69768	1.3	0.18452	2.0	0.01832

7.3 GAUSSIAN QUADRATURE FORMULAS

In the Gaussian quadrature formulas

$$\int_{-1}^{1} f(t)\,dt = \sum_{i=1}^{n} A_i f(t_i) + R_n(f) \tag{1}$$

the coefficients A_i and abscissas t_i $(i=1, 2, \ldots, n)$ are chosen so that the formula be accurate for all the polynomials of the highest possible degree N.

It is proved (see [1], [12]) that such numbers (A_i, t_i) are determined uniquely for $N=2n-1$. Table 53 contains the values of the abscissas t_i and coefficients A_i, as also the formulas of the remainder terms $R_n(f)$ for $n=4, 5, 7$.

Table 53

Abscissas t_i and Coefficients A_i of the Gaussian Quadrature Formulas

n	t_i	A_i	$R_n(f)$
4	$-t_1=t_4=0.861136312$ $-t_2=t_3=0.339981044$	$A_1=A_4=0.347854845$ $A_2=A_3=0.652145155$	$R_4(f)\approx 2.88\cdot 10^{-7} f^{(8)}(\xi)$ $-1<\xi<1$
5	$-t_1=t_5=0.906179846$ $-t_2=t_4=0.538469310$ $t_3=0$	$A_1=A_5=0.236926885$ $A_2=A_4=0.478628670$ $A_3=0.568888889$	$R_5(f)\approx 8.08\cdot 10^{-4} f^{(10)}(\xi)$ $-1<\xi<1$
7	$-t_1=t_7=0.949107912$ $-t_2=t_6=0.741531186$ $-t_3=t_5=0.405845151$ $t_4=0$	$A_1=A_7=0.129484966$ $A_2=A_6=0.279705391$ $A_3=A_5=0.381830051$ $A_4=0.417959184$	$R_7(f)\approx 2.13\cdot 10^{-15} f^{(14)}(\xi)$ $-1<\xi<1$

Inconvenience of application of the Gaussian quadrature formula consists in that the abscissas t_i and coefficients A_i, generally speaking, are irrational numbers. This shortcoming is compensated by its high accuracy with comparatively small number of integration points. The Gaussian formula is especially advantageous to be used when the integrand is rather complicated and much time is required to compute its values at each point of integration.

It is a very rare case to estimate successfully the error of the result for the Gaussian formulas, using the formula of the remainder term, since it is associated with computing high-order derivative of the integrand function (see Table 53). At present more convenient methods have been developed for practical use, which makes it possible to realize checking on accuracy (see [19]).

In computing the integral

$$\int_a^b f(x)\,dx$$

it is advisable to substitute the variable

$$x=\frac{b+a}{2}+\frac{b-a}{2}t.$$

Then Gauss' formula will have the form

$$\int_a^b f(x)\,dx=\frac{b-a}{2}\sum_{i=1}^{n}A_i f(x_i)+R_n^*(f), \tag{2}$$

where

$$x_i=\frac{b+a}{2}+\frac{b-a}{2}t_i, \tag{3}$$

$$R_n^*(f)=\left(\frac{b-a}{2}\right)^{2n+1}R_n(f). \tag{4}$$

Example 1. Compute the integral by Gauss' formula for $n=5$:

$$I=\int_0^1\frac{dx}{1+x^2}. \tag{5}$$

Solution. Let us substitute the variable

$$x=\frac{1}{2}+\frac{1}{2}t.$$

We get the integral

$$I=2\int_{-1}^{1}\frac{dt}{4+(t+1)^2}.$$

Table 54

Computing the Integral (5) by Gauss' Formula

i	t_i	$f(t_i)$	A_i
1	—0.906179846	0.24945107	0.236926885
2	—0.538469310	0.23735995	0.478628670
3	0	0.2	0.568888889
4	0.538469310	0.15706261	0.478628670
5	0.906179846	0.13100114	0.236926885

Tabulate the values of the integrand (Table 54) and then find the integral by Gauss' formula for $n=5$

$$I \approx 2[A_1 f(t_1) + A_2 f(t_2) + A_3 f(t_3) + A_4 f(t_4) + A_5 f(t_5)] = 0.78539816.$$

By way of comparison we give the exact value of $I = \frac{\pi}{4} = 0.785398163\ldots$. As is obvious, in the obtained result all the eight digits are correct. Note that already the sixth decimal digit of the result yielded by Simpson's formula in computing this integral ($h=0.1$) is erroneous.

Example 2. Compare the accuracy of evaluating the integrals

$$I_n = \int_{-1}^{1} |x|^n dx \quad (n=1, 2, 3, \ldots)$$

computed by the trapezoidal, Simpson's and Gauss' formulas for three points.

Solution. By the trapezoidal formula with the interval $h=1$ we have

$$I_n \approx 1\left(\frac{1}{2}+0+\frac{1}{2}\right)=1;$$

by Simpson's formula with the same interval we have

$$I_n \approx \frac{1}{3}(1+0+1)=\frac{2}{3}.$$

By Gauss' formula for $f(x)=|x|^n$ we get

$$I_n \approx \frac{5}{9}\left[f\left(-\sqrt{\frac{3}{5}}\right)+f\left(\sqrt{\frac{3}{5}}\right)\right]+\frac{8}{9}f(0)=\frac{10}{9}\left(\frac{3}{5}\right)^{n/2}.$$

The exact value of the integral is

$$I_n \approx 2\int_0^1 x^n dx = \frac{2}{n+1}.$$

Thus, for $n=1$ the trapezoidal formula yields the exact value $I_1=1$, Simpson's formula gives the error $\frac{1}{3}$, Gauss' formula yields the value $\frac{10}{9}\sqrt{\frac{3}{5}}=0.861$ with the error 0.139. For $n=2$ Simpson's and Gauss' formulas yield the exact value $I_2=\frac{2}{3}$, while the trapezoidal formula gives the error $\frac{1}{3}$. For $n=3$ the exact value is $I_3=\frac{2}{4}$, the Gaussian formula yields 0.5164 with the error 0.0164 Simpson's formula gives the error $\frac{2}{3}-\frac{2}{4}=0.167$, the trapezoidal formula gives the error 0.5, i.e. 100%. For $n=4$ the Gaussian formula yields the exact value $I_4=\frac{2}{5}$, Simpson's formula gives the error $\frac{2}{3}-\frac{2}{5}=\frac{4}{15}\approx 0.267$, the trapezoidal formula gives the error $\frac{3}{5}=0.6$.

This comparison shows that the trapezoidal formula can give a higher accuracy than Simpson's and Gauss' formulas for functions with discontinuities in the derivative. But for functions, which have a sufficient quantity of continuous derivatives, the Gaussian formula is considerably more accurate as compared with Simpson's formula, and the latter is more accurate than the trapezoidal formula. This is well illustrated by the following example employing three points of the integral

$$I=\int_{-1}^{1} e^x\,dx=e-e^{-1}=2.3504.$$

By the trapezoidal formula we get

$$I\approx\frac{0.3679}{2}+1+\frac{2.7183}{2}=2.5431 \text{ with the error } 0.193.$$

By Simpson's formula we get

$$I\approx\frac{1}{3}(0.3679+4\cdot 1+2.7133)=2.3621 \text{ with the error } 0.0117.$$

By Gauss' formula we get

$$I\approx\frac{5}{9}(0.4609+2.1686)+\frac{8}{9}=2.3497 \text{ with the error } 0.0007.$$

Problems

Using Gauss' formula, compute the following integrals and estimate the error (the remainder term R):

1. $\int_1^3 \frac{dx}{1+x}$ for $n=4$.

2. $\int_{-1}^{1} \frac{dx}{x+3}$ for $n=5$.

3. $\int_0^1 \sqrt{1+x}\,dx$ for $n=4$.

Compute the following integrals by means of Gauss' formula:

4. $\int_0^1 \frac{dt}{\sqrt{(t^2+1)(3t^2+4)}}$ for $n=4$.

5. $\int_0^1 \frac{\log(1+x)}{1+x^2}dx$ for $n=5$.

6. $\int_0^1 \frac{\ln(1+x)}{1+x^2}dx$ for $n=4$.

7. $\int_0^1 \frac{\ln(1+x)}{x}dx$ for $n=4$.

8. $I=\int_0^1 \frac{e^{-ax^2}dx}{1+\sin ax}$ for $n=5$, $a=0.5+0.2\cdot k$, $k=0, 1, \ldots, 10$.

9. $I=\int_0^{\pi/2} \sqrt{3-ax}\cos ax\,dx$ for $n=5$, $a=0.5+0.1\cdot k$, $k=0, 1, \ldots, 14$.

7.4 INTEGRATING WITH THE AID OF POWER SERIES

Consider the definite integral

$$\int_a^b f(x)\,dx. \tag{1}$$

Let the integrand $f(x)$ be expanded into a power series

$$f(x)=\sum_{k=0}^{\infty} c_k x^k,$$

converging on the interval $(-R, R)$, which contains the interval of integration $[a, b]$.

Applying the theorem on termwise integration of power series (see [55]), we can represent the integral (1) as a numerical series

$$\int_a^b f(x)\,dx = \sum_{k=0}^{\infty} \frac{c_k}{k+1}(b^{k+1} - a^{k+1}). \tag{2}$$

If the series (2) converges sufficiently rapid, then we can compute approximately the definite integral with the aid of the partial sum of the series

$$\int_a^b f(x)\,dx \approx \sum_{k=0}^{N} \frac{c_k}{k+1}(b^{k+1} - a^{k+1}). \tag{3}$$

The error of the final result is made up here of the following errors:

(1) the error due to the replacement of the series by the partial sum; this error (the truncation error) is equal to the remainder of the series;

(2) the errors due to rounding off in computing the sum (3).

For an alternating series with monotonically decreasing (in absolute value) terms the absolute value of the remainder term of the series does not exceed the absolute value of the first of the rejected terms of the series (see Examples 1 and 2). In other cases for estimating the remainder term of a series we use majorization of such numerical series whose remainder terms are readily estimated (see Example 3).

Example 1. Compute the integral

$$\int_0^1 e^{-x^2}\,dx,$$

by expanding the integrand into a power series and using seven terms of this expansion. Estimate the error.

Solution. We have

$$e^{-x^2} = 1 - x^2 + \frac{x^4}{2!} - \frac{x^6}{3!} + \ldots + (-1)^n \frac{x^{2n}}{n!} + \ldots .$$

This series converges for any x; integrating termwise the first seven terms, we get

$$\int_0^1 e^{-x^2}\,dx \approx \left[x - \frac{x^3}{3} + \frac{x^5}{5\cdot 2!} - \frac{x^7}{7\cdot 3!} + \frac{x^9}{9\cdot 4!} - \frac{x^{11}}{11\cdot 5!} + \frac{x^{13}}{13\cdot 6!}\right]_0^1 =$$

$$= 1 - \frac{1}{3} + \frac{1}{10} - \frac{1}{42} + \frac{1}{216} - \frac{1}{1320} + \frac{1}{9360}. \tag{4}$$

Estimating the remainder of the series, we have

$$|R_7| \leqslant \frac{x^{15}}{15 \cdot 7!}\Big|_0^1 = \frac{1}{75\,600} < 1.5 \cdot 10^{-5}.$$

Taking this into account, we compute the sum (4) to five decimal places (with one extra digit). We finally get

$$\int_0^1 e^{-x^2} dx = 0.7468$$

with all correct digits.

Example 2. Compute the integral

$$I = \int_0^{\pi/4} \sin(x^2)\, dx$$

to within 10^{-4} by expanding the integrand into a power series.

Solution. Expand the function $f(x) = \sin(x^2)$ into a power series

$$\sin(x^2) = x^2 - \frac{x^6}{3!} + \frac{x^{10}}{5!} - \frac{x^{14}}{x!} + \ldots .$$

This series converges for any value of x. Let us integrate it termwise on the interval from 0 to $\frac{\pi}{4}$:

$$I = \left[\frac{x^3}{3} - \frac{x^7}{7 \cdot 3!} + \frac{x^{11}}{11 \cdot 5!} - \frac{x^{15}}{15 \cdot 7!} + \ldots\right]_0^{\pi/4} =$$

$$= \frac{1}{3}\left(\frac{\pi}{4}\right)^3 - \frac{1}{7 \cdot 3!}\left(\frac{\pi}{4}\right)^7 + \frac{1}{11 \cdot 5!}\left(\frac{\pi}{4}\right)^{11} - \frac{1}{15 \cdot 7!}\left(\frac{\pi}{4}\right)^{15} + \ldots .$$

Since the obtained numerical series is an alternating one, it is sufficient to take the number of terms such that the first of the rejected terms be less than 10^{-4}. This condition is satisfied by the third term, since

$$\frac{\pi^{11}}{11 \cdot 5! 4^{11}} < \frac{(0.786)^{11}}{1320} = \frac{0.051}{1320} < 4 \cdot 10^{-5}.$$

Find the sum of the first two terms. Carry out the calculations with five decimal digits, and round off the final answer:

$$I = \frac{\pi^3}{3 \cdot 4^3} - \frac{\pi^7}{7 \cdot 3! 47} = \frac{1}{3}(0.78540)^3 \left[1 - \frac{(0.78540)^4}{14}\right] = 0.1571.$$

Example 3. Compute the integral

$$I = \int_0^{0.5} \frac{dx}{1 - x^5}$$

accurate to $\varepsilon = 10^{-4}$.

Solution. Expand the function $f(x)=\frac{1}{1-x^5}$ into a power series

$$\frac{1}{1-x^5}=1+x^5+x^{10}+x^{15}+\ldots+x^{5n-5}+\ldots.$$

The domain of convergence of this series is the interval $(-1, 1)$. The interval of integration is located within this interval, hence, the above written series may be integrated termwise.

Integrating on the interval $[0; 0.5]$, we get

$$I=\left[x+\frac{x^6}{6}+\frac{x^{11}}{11}+\frac{x^{16}}{16}+\ldots+\frac{x^{5n-4}}{5n-4}+\ldots\right]_0^{1/2}=$$
$$=\frac{1}{2}+\frac{1}{6\cdot 2^6}+\frac{1}{11\cdot 2^{11}}+\frac{1}{16\cdot 2^{16}}+\ldots+\frac{1}{(5n-4)\,2^{5n-4}}+\ldots.$$

Since already the third term is less than 10^{-4}, we try to take the sum of the first two terms as the approximate value of the integral, and estimate the sum of the rejected terms (the remainder of the series R_2):

$$R_2=\frac{1}{11\cdot 2^{11}}+\frac{1}{16\cdot 2^{16}}+\frac{1}{21\cdot 2^{21}}+\ldots+\frac{1}{(5n-4)\,2^{5n-4}}+\ldots.$$

Replace in all the terms, beginning with the second one, the factors 16, 21, and so on, of the denominator by the number 11. Then we get the sum of an infinitely decreasing geometric progression

$$R_2<\frac{1}{11\cdot 2^{11}}+\frac{1}{11\cdot 2^{16}}+\ldots+\frac{1}{11\cdot 2^{5n-4}}+\ldots=\frac{1}{11\cdot 2^{11}}\cdot\frac{1}{1-\frac{1}{2^5}}=$$
$$=\frac{1}{11\cdot 2^6\cdot 31}=\frac{1}{21\,824}<\frac{1}{2}\cdot 10^{-4}.$$

Thus, the sum of the first two terms of the integrated series yields the value of the integral with the specified accuracy.

We get the answer:

$$\int_0^{0,5}\frac{dx}{1-x^5}\approx\frac{1}{2}+\frac{1}{6\cdot 2^6}=0.5026.$$

Problems

In Problems 1 to 4 compute the integrals by expanding the integrand into a power series and taking three terms of this expansion. Estimate the error R.

1. $\int_0^{0.1}\frac{\ln(1+x)}{x}dx.$

2. $\int_0^1\cos(x^2)\,dx.$

3. $\int_0^{0.25} \sqrt{x}\, e^{-\sqrt{x}}\, dx.$ 4. $\int_0^{0.5} x^2 \sqrt{1+x^2}\, dx.$

Compute the integrals with the required accuracy ε.

5. $\int_0^2 \frac{\sin x}{x} dx, \quad \varepsilon = 10^{-5}.$ 6. $\int_0^{\pi} \frac{\sin x}{x} dx, \quad \varepsilon = 10^{-3}.$

7. $\int_0^1 \frac{\ln(1+\sqrt{x})}{\sqrt[3]{x}} dx, \quad \varepsilon = 10^{-4}.$

8. Compute

$$\arcsin 0.2 = \int_0^{0.2} \frac{dx}{\sqrt{1-x^2}}$$

to within 10^{-5}.

Expanding the integrand into a series, compute the following integrals to within 10^{-3}:

9. $\int_0^1 \frac{\sinh x}{x} dx.$ 10. $\int_0^{1/3} \frac{dx}{\sqrt[3]{1-x^3}}.$ 11. $\int_0^1 \frac{dx}{\sqrt{1+x^4}}.$

12. $\int_0^{0.5} \frac{\arctan x}{x} dx.$ 13. $\int_0^{0.5} \frac{\arcsin x}{x} dx.$

Compute the following integrals to within 10^{-4}:

14. $\int_0^{\sqrt{3}/3} x^3 \arctan x\, dx.$ 15. $\int_0^1 \sqrt[3]{x} \cos x\, dx.$

16. $\int_0^1 \frac{\sin x}{\sqrt{x}} dx.$ 17. $\int_0^{0.25} \ln(1+\sqrt{x})\, dx.$

18. $\int_{0.1}^{0.2} \frac{e^{-x}}{x^3} dx.$ 19. $\int_0^{1/9} \sqrt{x}\, e^x dx.$ 20. $\int_{0.2}^{0.3} \frac{dx}{\sqrt[3]{1+x^2}}.$

21. $\int_{0.5}^{0.6} \frac{dx}{\sqrt{1-x^4}}.$ 22. $\int_0^{0.5} \sqrt{1+x^5}\, dx.$ 23. $\int_0^{0.5} \sqrt{x-x^3}\, dx.$

24. $\int_0^1 \frac{\sin^2 x}{\sqrt{x}} dx.$ 25. $\int_{0.5}^{0.8} \sqrt{1+x^3}\, dx.$ 26. $\int_{0.5}^1 \sqrt{1+x^3}\, dx.$

7.5 INTEGRALS OF DISCONTINUED FUNCTIONS. THE METHOD OF KANTOROVICH FOR ISOLATING SINGULARITIES

Suppose it is required to compute the improper integral

$$\int_a^b f(x)\,dx, \tag{1}$$

where the integrand function $f(x)$ turns into infinity at some point c of the interval $[a, b]$. As is known, by definition, we put

$$\int_a^b f(x)\,dx = \lim_{\substack{\delta_1 \to 0 \\ \delta_2 \to 0}} \left\{ \int_a^{c-\delta_1} f(x)\,dx + \int_{c+\delta_2}^b f(x)\,dx \right\}. \tag{2}$$

To evaluate the converging improper integral to the preassigned accuracy ε choose the positive number δ_1 and δ_2 so small as to fulfill the inequality

$$\left| \int_{c-\delta_1}^{c+\delta_2} f(x)\,dx \right| < \frac{\varepsilon}{2}.$$

Then, using some quadrature formulas, we compute approximately the definite integrals

$$\int_a^{c-\delta_1} f(x)\,dx \quad \text{and} \quad \int_{c+\delta_2}^b f(x)\,dx. \tag{3}$$

If S_1 and S_2 are approximate values of the integrals (3) accurate to $\frac{\varepsilon}{4}$ each, then

$$\int_a^b f(x)\,dx \approx S_1 + S_2$$

accurate to ε.

Example 1. Compute the integral to within 0.05

$$I = \int_{0.3}^{2} \frac{e^{-x}\,dx}{\sqrt[4]{2+x-x^2}}.$$

Solution. The integrand has a discontinuity at $x=2$. Let us present the integral as a sum of two integrals

$$I_1 = \int_{0.3}^{2-\delta} \frac{e^{-x}\,dx}{\sqrt[4]{2+x-x^2}}, \quad I_2 = \int_{2-\delta}^{2} \frac{e^{-x}\,dx}{\sqrt[4]{2+x-x^2}}$$

and choose δ so that the quantity I_2 turns out to be sufficiently small. For instance, for $\delta \leqslant 0.1$ the integral I_2 satisfies the condition

$$0 < I_2 < \frac{e^{-1.9}}{\sqrt[4]{2.9}} \int_{2-\delta}^{2} \frac{dx}{\sqrt[4]{2-x}} = 0.115 \cdot \frac{4}{3} \delta^{3/4} = 0.153\delta^{3/4}.$$

Putting $\delta = 0.1$, we get $I_2 < 0.028$.

Taking into account the estimate for the integral I_2, let us compute

$$I_1 = \int_{0.3}^{1.9} \frac{e^{-x}\,dx}{\sqrt[4]{2+x-x^2}}$$

by Simpson's formula to within 0.022. Computations for the intervals $2h = 0.8$ and $h = 0.4$ yield

$$I_{1;\,2h} = 0.519, \quad I_{1;\,h} = 0.513.$$

Hence, we may write to within 0.004

$$I_1 \approx 0.51.$$

Thus, putting $I \approx I_1$, we get

$$I \approx 0.51,$$

the total error of the final result satisfying the given condition

1. The method of Kantorovich for isolating singularities. In many cases a useful device in calculating the approximate value of an integral with finite limits of a discontinuous function is the method of L. V. Kantorovich for isolating singularities (see [1], [12], [21], [27]). The underlying principle of this method is that we take out of the integrand $f(x)$ a certain function $g(x)$ having the same singularities as $f(x)$, which is integrable in elementary terms on the given interval $[a, b]$ and is such that the difference $f(x) - g(x)$ has a necessary number of derivatives. Write the integral in the form

$$\int_a^b f(x)\,dx = \int_a^b g(x)\,dx + \int_a^b [f(x) - g(x)]\,dx.$$

The first integral is taken directly, and the second integral is readily evaluated by standard quadrature formulas.

The function $g(x)$ is chosen in a variety of ways depending on the concrete case. Let us deduce the rule of forming such a function for one widely used class of integrals. Let the integrand function have the form

$$f(x) = (x-c)^{\alpha} \varphi(x), \quad a \leqslant c \leqslant b, \tag{4}$$

where $-1 < \alpha < 0$, and $\varphi(x)$ is continuous and has a sufficient number of derivatives on the interval $[a, b]$. Then $f(x)$ can be written in the form

$$f(x) = \left[\varphi(c)(x-c)^{\alpha} + \frac{\varphi'(c)}{1!}(x-c)^{\alpha+1} + \frac{\varphi''(c)}{2!}(x-c)^{\alpha+2} + \ldots \right.$$
$$\left. \ldots + \frac{\varphi^{(k)}(c)}{k!}(x-c)^{\alpha+k}\right] + (x-c)^{\alpha}\left[\varphi(x) - \varphi(c) - \right.$$
$$\left. - \frac{\varphi'(c)}{1!}(x-c) - \frac{\varphi''(c)}{2!}(x-c)^2 - \ldots - \frac{\varphi^{(k)}(c)}{k!}(x-c)^k\right]. \quad (5)$$

The first square brackets contain a power function which is integrated directly. The second squared expression vanishes at $x=c$ together with its derivatives up to order k included. The product of this expression by the factor $(x-c)^{\alpha}$ is a function continuous together with the derivatives up to order $k-1$. Therefore, the integral of this function can be evaluated by one of the standard quadrature formulas.

Example 2. Calculate the approximate value of the integral

$$I = \int_0^{0.5} \frac{dx}{\sqrt{x(1-x)}}.$$

Solution. The integrand has a discontinuity at $x=0$. Let us rewrite it as

$$f(x) = x^{-1/2}(1-x)^{-1/2}.$$

Thus $\alpha = -1/2$, $c=0$, $\varphi(x) = (1-x)^{-1/2}$. By Taylor's formula we have

$$\varphi(x) = 1 + \frac{1}{2}x + \frac{3}{8}x^2 + \frac{5}{16}x^3 + \frac{35}{128}x^4 + R_4(x).$$

Then $f(x)$ can be written in the form

$$f(x) = \left[x^{-1/2} + \frac{1}{2}x^{1/2} + \frac{3}{8}x^{3/2} + \frac{5}{16}x^{5/2} + \frac{35}{128}x^{7/2}\right] + \frac{\psi(x)}{\sqrt{x}},$$

where

$$\psi(x) = \frac{1}{\sqrt{1-x}} - \left(1 + \frac{1}{2}x + \frac{3}{8}x^2 + \frac{5}{16}x^3 + \frac{35}{128}x^4\right),$$

and $\psi(0) = 0$.

Hence

$$I = \int_0^{0.5}\left(x^{-1/2} + \frac{1}{2}x^{1/2} + \frac{3}{8}x^{3/2} + \frac{5}{16}x^{5/2} + \frac{35}{128}x^{7/2}\right)dx + I_1 =$$
$$= 1.5691585 + I_1,$$

where

$$I_1 = \int_0^{0.5} \frac{\psi(x)}{\sqrt{x}}\, dx.$$

Evaluate the integral I_1 by Simpson's formula for $n = 10$, i.e. $h = 0.05$. Table 55 gives the values of the integrand accurate

Table 55

Values of the Function $y = \dfrac{\psi(x)}{\sqrt{x}}$

i	x_i	y_i	m_i
0	0	0.000000	1
1	0.05	0.000000	4
2	0.10	0.000009	2
3	0.15	0.000056	4
4	0.20	0.000216	2
5	0.25	0.000624	4
6	0.30	0.001508	2
7	0.35	0.003225	4
8	0.40	0.006316	2
9	0.45	0.011588	4
10	0.50	0.020239	1
		Σ	0.098309

to 10^{-6}. The last row of the table presents the sum of the products of y_i by m_i, whence

$$I_1 = \frac{0.05}{3} \cdot 0.098309 = 0.0016385.$$

We get finally

$$I = \int_0^{0.5} \frac{dx}{\sqrt{x(1-x)}} = 1.5691585 - 0.0016385 = 1.5707970.$$

For the sake of comparison we give the exact value of the integral $I = \frac{\pi}{2} = 1.5707963$.

Note. The Kantorovich method is also applicable to improper integrals of discontinuous functions of another type (see [1], [21]). If the integrand function has several points of discontinuity, then to evaluate an integral it is sufficient to subdivide the interval of integration into parts, each containing only one singular point

of the integrand function, and take the advantage of the additive property of the integral. This method may turn out to be also helpful in computing proper integrals if the integrand, though being a continuous function, does not have a sufficient number of continuous derivatives (which makes it difficult to estimate the errors).

2. **Applying the weight quadrature formulas.** Consider the improper integral of a discontinuous function

$$\int_a^b f(x)\,dx.$$

Represent the integrand in the form of a product of two functions $\varphi(x)$ and $p(x)$:

$$f(x)=\varphi(x)\,p(x), \tag{6}$$

$\varphi(x)$ being bound on the interval $[a, b]$ and having a sufficient number of derivatives, while $p(x) > 0$ on $[a, b]$. Then it becomes possible to choose a quadrature formula of the form

$$\int_a^b p(x)\varphi(x)\,dx \approx \sum_{k=1}^{n} C_k^{(n)}\varphi(x_k), \tag{7}$$

in which the free coefficients $C_k^{(n)}$ are independent of $\varphi(x)$, and the abscissas x_k are determined so that the formula becomes accurate for polynomials of a highest possible degree. The possibility of constructing such formulas is shown in [1], [21]. The function $p(x)$ is called the *weight function* or the *weight*. The sense of applying the quadrature formulas with the weight $p(x)$ for integrating discontinuous functions consists in that the remainder term is independent of $p(x)$, i.e. of the discontinuous portion of the function.

Supplied in [19], [22] are the values of the coefficients $C_k^{(n)}$ and the abscissas x_k of the quadrature formulas of the form (7) for various weight functions $p(x)$.

Table 56 contains the values of $C_k^{(n)}$ and x_k for the case when $a=0$, $b=1$, and the weight function has the form $p(x)=x^{-1/2}$.

For $a=-1$, $b=1$ and $p(x)=\dfrac{1}{\sqrt{1-x^2}}$ there occurs the *Hermitian quadrature formula* (see [1])

$$\int_{-1}^{1} \frac{\varphi(x)\,dx}{\sqrt{1-x^2}} \approx \frac{\pi}{n}\sum_{k=1}^{n}\varphi(x_k), \tag{8}$$

where $x_k=\cos\dfrac{2k-1}{2n}\pi$ and

$$R(f)=\frac{\pi}{(2n)!\,2^{n-1}}f^{(2n)}(\xi), \quad -1<\xi<1. \tag{9}$$

Table 56

Values of x_k and $C_k^{(n)}$ for the Quadrature Formula

$$\int_0^1 \frac{\varphi(x)}{\sqrt{x}}\,dx \approx \sum_{k=1}^{n} C_k^{(n)} \varphi(x_k)$$

n	k	x_k	$C_k^{(n)}$
3	1	0.056939	0.935828
	2	0.437198	0.721523
	3	0.869499	0.342649
4	1	0.033648	0.725368
	2	0.276184	0.627413
	3	0.634677	0.444762
	4	0.922157	0.202457
5	1	0.022164	0.591048
	2	0.187831	0.538533
	3	0.461597	0.438173
	4	0.748335	0.298903
	5	0.948494	0.133343

Example 3. Compute the approximate value of the integral

$$I = \int_{-1}^{1} \frac{dx}{\sqrt{1-x^4}}.$$

Solution. Let us represent the integrand in the form

$$\frac{1}{\sqrt{1-x^4}} = \frac{1}{\sqrt{1-x^2}} \cdot \frac{1}{\sqrt{1+x^2}}$$

and consider

$$p(x) = \frac{1}{\sqrt{1-x^2}}$$

as a weight function. Then the given integral can be computed by the Hermitian quadrature formula (8)

$$I = \int_{-1}^{1} \frac{dx}{\sqrt{1-x^4}} \approx \frac{\pi}{n} \sum_{k=1}^{n} \frac{1}{\sqrt{1+x_k^2}}.$$

Putting $n = 6$, we get

$$I \approx \frac{\pi}{6}\left[\frac{2}{\sqrt{1+\cos^2 15^\circ}} + \frac{2}{\sqrt{1+\cos^2 45^\circ}} + \frac{2}{\sqrt{1+\cos^2 75^\circ}}\right] = 2.221329.$$

Let us note that the direct computation of the integral with six correct digits yields

$$I = 2.221441.$$

Example 4. Compute the approximate value of the integral

$$I = \int_0^1 \frac{dx}{(4-x)\sqrt{x}}.$$

Solution. Put

$$p(x) = \frac{1}{\sqrt{x}}, \quad \varphi(x) = \frac{1}{4-x}.$$

Using the values of x_k and $C_k^{(n)}$ listed in Table 56 for $n=4$, we get by formula (7)

$$I = \int_0^1 \frac{dx}{(4-x)\sqrt{x}} \approx \frac{0.7254}{3.9664} + \frac{0.6274}{3.7238} + \frac{0.4448}{3.3653} + \frac{0.2025}{3.0778} = 0.5493.$$

Direct evaluation of the integral yields

$$\int_0^1 \frac{dx}{(4-x)\sqrt{x}} = \frac{1}{2}\ln 3 \approx 0.5493.$$

Problems

Using the Kantorovich method for isolating singularities, calculate the approximate value of the following integrals.

1. $\int_0^1 \frac{dx}{\sqrt{x}\sqrt[4]{(1-x)^3}}$ accurate to 10^{-5}.
2. $\int_0^1 \frac{dx}{(1+x)\sqrt{x}}$ accurate to 10^{-6}.
3. $\int_0^2 \frac{dx}{\sqrt{x}\,(e^{x/2}+3)}$ accurate to 10^{-5}.
4. $\int_0^1 \frac{x^2\,dx}{\sqrt{1-x^2}}$ accurate to 10^{-6}.
5. $\int_1^2 \frac{dx}{x\sqrt{x^2-1}}$ accurate to 10^{-6}.

Using the weight quadrature formulas, compute the following integrals

6. $\int_{-1}^{1} \frac{e^{2x}}{\sqrt{1-x^2}}\,dx$ for $n=5$.

7. $\int_{0}^{1} \frac{dx}{(1+x)\sqrt{x}}$ for $n=5$.

8. $\int_{0}^{1} \frac{dx}{\sqrt{x}\,(e^{0.4x}+1.5)}$ for $n=4$.

9. $\int_{-1}^{1} \frac{dx}{(1+x^2)\sqrt{1-x^2}}$ for $n=5$.

10. $\int_{0}^{1} \frac{e^{a(x-1)}}{\sqrt{x}\,(x+b)}\,dx$ for $n=5$, $a=0.60+0.07k$, $k=0, 1, \ldots, 8$, $b=2.00+0.25\cdot k$, $k=0, 1, 2, \ldots, 6$.

11. $I=\int_{-1}^{1} \frac{\cos ax}{(0.3+x^2)\sqrt{1-x^2}}\,dx$ for $n=12$, $a=2.6+0.04\cdot k$, $k=0, 1, 2, \ldots, 10$.

7.6 INTEGRALS WITH INFINITE LIMITS

1. Truncation method. To evaluate the convergent improper integral

$$\int_{a}^{\infty} f(x)\,dx$$

to a given accuracy ε, we represent it in the form

$$\int_{a}^{\infty} f(x)\,dx = \int_{a}^{b} f(x)\,dx + \int_{b}^{\infty} f(x)\,dx,$$

where b is chosen so large that the inequality

$$\left|\int_{b}^{\infty} f(x)\,dx\right| < \frac{\varepsilon}{2}$$

is valid.

Then the proper integral

$$\int_a^b f(x)\,dx$$

may be computed from one of the quadrature formulas to an accuracy of $\frac{\varepsilon}{2}$. Then we approximately put

$$\int_a^\infty f(x)\,dx \approx \int_a^b f(x)\,dx.$$

Example 1. Find the approximate value of the integral

$$\int_2^\infty \frac{dx}{1+x^3}$$

to within $\varepsilon = 10^{-2}$.

Solution. Choose the number b so that the inequality

$$\int_b^\infty \frac{dx}{1+x^3} < \frac{10^{-2}}{2}$$

is fulfilled.

Noting that

$$\int_b^\infty \frac{dx}{1+x^3} < \int_b^\infty \frac{dx}{x^3} = \frac{1}{2b^2},$$

we choose b from the condition

$$\frac{1}{2b^2} = \frac{10^{-2}}{2},$$

whence $b = 10$. We approximately put

$$\int_2^\infty \frac{dx}{1+x^3} \approx \int_2^{10} \frac{dx}{1+x^3} = I$$

and evaluate the definite integral obtained by Simpson's formula to within $\frac{1}{2} \cdot 10^{-2}$, putting $h_1 = 1$, $h_2 = 2$.

The results of computations are tabulated below. The last row of the table are the sums $\sum y_k m_k$ and $\sum y_k m'_k$ by which approximate values of the integral are computed. For $h_1 = 1$ we get $I_1 = \frac{1}{3} \cdot 0.3477 = 0.1159$ for $h_2 = 2$ we obtain $I_2 = \frac{2}{3} \cdot 0.1809 = 0.1206$, the difference between the two values being less than $\frac{1}{2} \cdot 10^{-2}$.

Table 57

Computing Integral I by Simpson's Formula

k	x_k	$1+x^3$	y_k	m_k	m_k'
0	2.0	9	0.1111	1	1
1	3.0	28	0.0357	4	
2	4.0	65	0.0154	2	4
3	5.0	126	0.0079	4	
4	6.0	217	0.0046	2	2
5	7.0	344	0.0029	4	
6	8.0	513	0.0020	2	4
7	9.0	730	0.0014	4	
8	10.0	1001	0.0010	1	1
			Σ	0.3477	0.1809

Finally we have

$$\int_2^{\infty} \frac{dx}{1+x^3} = 0.12.$$

2. Applying the weight quadrature formulas. In evaluating integrals

$$\int_a^{\infty} p(x)\,\varphi(x)\,dx$$

it is often convenient to use quadrature formulas of the form

$$\int_a^{\infty} p(x)\,\varphi(x)\,dx \approx \sum_{k=1}^{n} A_k^{(n)} \varphi(x_k)$$

in which the coefficients $A_k^{(n)}$ are independent of $\varphi(x)$ and the abscissas x_k are chosen so as to make the formula accurate for polynomials of a highest possible degree.

For $p(x) = e^{-x^2}$ there exist *Chebyshev-Hermit weight quadrature formula* (see [21], [22], [33]).

$$\int_{-\infty}^{\infty} e^{-x^2} \varphi(x)\,dx \approx \sum_{k=1}^{n} A_k^{(n)} \varphi(x_k), \tag{1}$$

exact for polynomials of degree not higher than $2n-1$. The remainder term of formula (1) has the form

$$R_n(\varphi) = \frac{n!\sqrt{\pi}}{2^n (2n)!} \varphi^{(2n)}(\xi). \tag{2}$$

Table 58

Values of $A_k^{(n)}$ and x_k for Chebyshev-Hermit Weight Quadrature Formula

n	x_k	$A_k^{(n)}$
3	$-x_1=x_3=1.224745$ $x_2=0$	$A_1=A_3=0.295409$ $A_2=1.181636$
4	$-x_1=x_4=1.650680$ $-x_2=x_3=0.524648$	$A_1=A_4=0.081313$ $A_2=A_3=0.804914$
5	$-x_1=x_5=2.020183$ $-x_2=x_4=0.958572$ $x_3=0$	$A_1=A_5=0.019953$ $A_2=A_4=0.393619$ $A_3=0.945309$

Table 58 presents the values of the coefficients $A_k^{(n)}$ and abscissas x_k for certain n.

For $p(x)=e^{-x}$ there exist the *Chebyshev-Lagerr weight quadrature formulas* (see [21], [22], [33])

$$\int_0^\infty e^{-x}\varphi(x)\,dx \approx \sum_{k=1}^{n} A_k^{(n)}\varphi(x_k), \tag{3}$$

which are exact for polynomials of degree not higher than $2n-1$. The remainder term of formula (3) has the form

$$R_n(\varphi)=\frac{n!\,\Gamma(n+1)}{(2n)!}\varphi^{(2n)}(\xi),$$

where $\Gamma(n+1)$ is a gamma function. Tabulated below are the values of $A_k^{(n)}$ and x_k for certain n.

Example 2. Evaluate the integral

$$\int_{-\infty}^{\infty} e^{-x^2}\cos x\,dx$$

using the quadrature formula (1) for $n=5$.

Solution. Since the abscissas of the quadrature formula are situated symmetrically about $x=0$, and the coefficients corresponding to the symmetric abscissas are equal, by formula (1) we have

$$\int_{-\infty}^{\infty} e^{-x^2}\cos x\,dx \approx A_3\cos x_3+2(A_4\cos x_4+A_5\cos x_5)=0.945309+$$
$$+2(0.393619\cdot 0.574689-0.019953\cdot 0.434413)=1.380390.$$

Table 59

Values of $A_k^{(n)}$ and x_k for Chebyshev-Lagerr Weight Quadrature Formulas

n	k	x_k	$A_k^{(n)}$
3	1	0.415774	0.711093
	2	2.294280	0.278518
	3	6.289945	0.010389
4	1	0.322548	0.603154
	2	1.745761	0.357419
	3	4.536620	0.038888
	4	9.395071	0.000539
5	1	0.263560	0.521756
	2	1.413403	0.398667
	3	3.596425	0.075942
	4	7.085810	0.003612
	5	12.640801	0.000023

For comparison let us note that the exact value of the integral is equal to

$$\sqrt{\pi}e^{-1/4} = 1.3803885.$$

Problems

Compute to the indicated accuracy ε the following improper integral by the truncation method:

1. $\int_1^\infty \frac{\arctan x}{1+x^3}\,dx$, $\varepsilon = 10^{-2}$. 2. $I = \int_1^\infty \frac{xe^{-x^2}}{2+\sin x}\,dx$, $\varepsilon = 10^{-5}$.

3. $I = \int_5^\infty \frac{dx}{\sqrt{1+ax^5}}$, $\varepsilon = 10^{-3}$, $a = 0.5 + 0.1 \cdot k$, $k = 0, 1, 2, \ldots, 10$.

4. $\int_1^\infty \frac{e^{-x}}{a+x}\,dx$, $\varepsilon = 10^{-5}$, $a = 0.4 + 0.2 \cdot k$, $k = 0, 1, 2, \ldots, 12$.

Compute the following integrals using the weight quadrature

formulas:

5. $\int_0^\infty \frac{xe^{-x}}{x+2}dx$, $n=4$. 6. $\int_{-\infty}^{\infty} \frac{x+1}{x+2}e^{-x^2}dx$, $n=4$.

7. $I=\int_0^\infty \frac{\sin ax}{a+x}e^{-x}dx$, $n=5$, $a=0.5+0.1\cdot k$, $k=0, 1, 2, \ldots, 10$.

8. $I=\int_0^\infty \frac{e^{-x}}{a+\sqrt{x}}dx$, $n=5$, $a=0.6+0.2\cdot k$, $k=0, 1, 2, \ldots, 10$.

9. $I=\int_{-\infty}^{\infty} \frac{e^{-x^2}}{\sqrt{2+\sin ax}}dx$, $n=5$, $a=1.5+0.2\cdot k$, $k=0, 1, 2, \ldots, 10$.

10. $I=\int_{-\infty}^{\infty} \frac{\cos ax}{a+x^2}e^{-x^2}dx$, $n=5$, $a=1.0+0.2\cdot k$, $k=0, 1, 2, \ldots, 10$.

7.7 MULTIPLE INTEGRALS. THE METHOD OF DOUBLE INTEGRATION. THE METHOD OF LYUSTERNIK AND DITKIN. THE MONTE CARLO METHOD.

1. The method of repeated application of quadrature formulas (see [1], [11], [12]). Let it be required to evaluate the double integral

$$I=\iint_G f(x, y)\,dx\,dy, \tag{1}$$

where the domain G represents a rectangle

$$a\leqslant x\leqslant b, \quad c\leqslant y\leqslant d.$$

Evaluation of such an integral is reduced to double integration, i.e.

$$I=\int_a^b dx\int_c^d f(x, y)\,dy. \tag{2}$$

Denote

$$\int_c^d f(x, y)\,dy=F(x)$$

and for computing the integral

$$\int_a^b F(x)\,dx \tag{3}$$

let us apply Simpson's formula with spacing h with respect to x:

$$\int_a^b F(x)\,dx \approx \frac{h}{3}[F_0+F_n+2(F_2+F_4+\ldots+F_{n-2})+ \\ +4(F_1+F_3+\ldots+F_{n-1})],$$

where $h=\frac{b-a}{n}$ (n is even),

$$F_i=F(x_i)=\int_c^d f(x_i, y)\,dy,\ x_i=a+ih\ (i=0, 1, \ldots, n). \qquad (4)$$

Thus, the problem is reduced to evaluating $n+1$ integrals of the form (4). The considered method is the simplest method of approximate computation of multiple integrals. It is readily programmed when carrying out calculations on a computer, since it allows the operator to make use of available standard programmes for computing single integrals. In evaluating integrals (3) and (4) we may apply various quadrature formulas.

Multiple integrals of degree $n>2$ can be computed in an analogous way.

The shortcomings of the method:

(1) it is suitably used only for rectangular domains of integration;

(2) with an increase in multiplicity of an integral the amount of computational work increases abruptly;

(3) a higher accuracy obtained on account of reduced integrals of integration considerably increases amount of calculations.

Example 1. Evaluate the approximate value of the integral

$$\int_0^{\pi/2} dx \int_0^{\pi/4} \sin(x+y)\,dy.$$

Solution. Let us introduce the following notation:

$$\int_0^{\pi/4} \sin(x+y)\,dy = F(x).$$

Then by Simpson's formula we have for $n=4$

$$\int_0^{\pi 2} F(x)\,dx \approx \frac{\pi}{24}[F_0+F_4+2F_2+4(F_1+F_3)]. \qquad (5)$$

The approximate values of the integrals

$$F_i=F(x_i)=\int_0^{\pi/4} \sin(x_i+y)\,dy \quad \left(x_i=\frac{\pi}{8}i,\quad i=0,\ 1,\ 2,\ 3,\ 4\right)$$

are computed by Simpson's formula for $n=2$. We successively get

$$F_0=\int_0^{\pi/4}\sin y\,dy\approx\frac{\pi}{24}[\sin y_0+4\sin y_1+\sin y_2]=$$

$$=\frac{\pi}{24}\left[\sin 0+4\sin\frac{\pi}{8}+\sin\frac{\pi}{4}\right]=\frac{\pi}{24}(0+4\cdot 0.3827+0.7071)=$$

$$=\frac{\pi}{24}\cdot 2.2379,$$

$$F_1=\int_0^{\pi/4}\sin\left(\frac{\pi}{8}+y\right)dy\approx\frac{\pi}{24}\left(\sin\frac{\pi}{8}+4\sin\frac{\pi}{4}+\sin\frac{3\pi}{8}\right)=\frac{\pi}{24}\cdot 4.1350,$$

$$F_2=\int_0^{\pi/4}\sin\left(\frac{\pi}{4}+y\right)dy\approx\frac{\pi}{24}\left(\sin\frac{\pi}{4}+4\sin\frac{3\pi}{8}+\sin\frac{\pi}{2}\right)=\frac{\pi}{24}\cdot 5.4027,$$

$$F_3=\int_0^{\pi/4}\sin\left(\frac{3\pi}{8}+y\right)dy\approx\frac{\pi}{24}\left(\sin\frac{3\pi}{8}+4\sin\frac{\pi}{2}+\sin\frac{5\pi}{8}\right)=\frac{\pi}{24}\cdot 5.8478,$$

$$F_4=\int_0^{\pi/4}\sin\left(\frac{\pi}{2}+y\right)dy\approx\frac{\pi}{24}\left(\sin\frac{\pi}{2}+4\sin\frac{5\pi}{8}+\sin\frac{3\pi}{4}\right)=\frac{\pi}{24}\cdot 5.4027.$$

Substituting the values of $F_i\,(i=0,\ 1,\ 2,\ 3,\ 4)$ into formula (5), we obtain

$$I=\int_0^{\pi/2}\int_0^{\pi/4}\sin(x+y)\,dx\,dy\approx\left(\frac{\pi}{24}\right)^2\cdot[2.2379+5.4027+2\cdot 5.4027+$$

$$+4\,(4.1350+5.8478)]=0.01713473\cdot 58.3772=1.00028.$$

For the sake of comparison we give the exact value of the integral $I=1$.

2. Method of L. A. Lyusternik and V. A. Ditkin (see [1], [11], [24]). For evaluating the double integral

$$\iint_G f(x,\ y)\,dx\,dy$$

the following approximate formulas are used

$$\iint_G f(x,\ y)\,dx\,dy\approx\sum_{i=1}^{n}c_i f(x_i,\ y_i),\qquad(6)$$

where the coefficients c_i and points $M_i(x_i,\ y_i)$ are chosen so as to make formula (6) accurate for polynomials of some sufficiently high degree with a minimum of points M_i.

If the domain G is a unit circle with the origin as centre, then the integral is computed by the *Lyusternik-Ditkin cubature*

formula (see [24])

$$\iint\limits_G f(x, y)\,dx\,dy \approx \pi\left[\frac{1}{4} f(0) + \frac{1}{8}\sum_{i=0}^{5} f(M_i)\right], \tag{7}$$

where the points M_i have the following polar coordinates

$$\rho_i = \sqrt{\frac{2}{3}}, \quad \varphi_i = \frac{\pi}{3} i \quad (i = 0, 1, 2, 3, 4, 5),$$

i.e. points M_i are the vertices of a regular hexagon inscribed in a circle of radius $\sqrt{\frac{2}{3}}$.

If the domain G has the form of a regular hexagon inscribed in the unit circle, then the following *Lyusternik-Ditkin cubature formula* takes place (see [24]):

$$\iint\limits_G f(x, y)\,dx\,dy \approx \frac{\sqrt{3}}{2}\left[\frac{43}{56} f(0) + \frac{125}{336}\sum_{i=0}^{5} f(M_i)\right], \tag{8}$$

where points M_i have the following polar coordinates

$$\rho_i = \frac{\sqrt{14}}{15}, \quad \varphi_i = \frac{\pi}{3} i \quad (i = 0, 1, 2, 3, 4, 5).$$

If the domain G is a square

$$-1 \leqslant x \leqslant 1, \quad -1 \leqslant y \leqslant 1,$$

then the cubature formula has the form (see [7])

$$\begin{aligned}\int_{-1}^{1}\int_{-1}^{1} f(x, y)\,dx\,dy \approx \frac{8}{7} f(0, 0) + \frac{20}{63}\left[f\left(\sqrt{\frac{14}{15}}, 0\right) + \right.\\ \left. + f\left(-\sqrt{\frac{14}{15}}, 0\right)\right] + \frac{5}{9}\left[f\left(\sqrt{\frac{1}{3}}, \sqrt{\frac{3}{5}}\right) + \right.\\ + f\left(-\sqrt{\frac{1}{3}}, \sqrt{\frac{3}{5}}\right) + f\left(-\sqrt{\frac{1}{3}}, -\sqrt{\frac{3}{5}}\right) + \\ \left. + f\left(\sqrt{\frac{1}{3}}, -\sqrt{\frac{3}{5}}\right)\right].\end{aligned} \tag{9}$$

Note 1. Formulas (7) to (9) are applicable to domains of the form:
(1) a circle of arbitrary radius,
(2) a regular hexagon inscribed in a circle of arbitrary radius,
(3) an ellipse,
(4) a rectangle,
if an appropriate substitution of variables is performed.

A domain of arbitrary configuration can be approximately replaced by a sum of above indicated domains of the standard type.

Note 2. Errors are estimated in [7] and [24]. These works show that the Lyusternik-Ditkin formulas require considerably less amount of computations

as compared with the formulas of the previous subsection ensuring the same accuracy.

But the coefficients c_i and coordinates of points $M_i(x_i, y_i)$ are calculated comparatively easily only for rather simple domains with a small number of points M_i.

For estimating the errors we can use double calculation when dealing with domain of the same type but a different size.

Example 2. Evaluate the integral

$$\iint\limits_G \sqrt{x^2+y^2}\,dx\,dy \tag{10}$$

over the domain G bounded by the circle $x^2+y^2=2x$.

Solution. Rewrite the equation of the given circle in the form

$$(x-1)^2+y^2=1$$

and make the change of variables

$$x_1=x-1, \quad y_1=y.$$

Then the integral (10) will turn into the integral

$$I=\iint\limits_{G_1} \sqrt{(x_1+1)^2+y_1^2}\,dx_1\,dy_1,$$

where G_1 is the unit circle $x_1^2+y_1^2\leqslant 1$.

To evaluate the integral I take advantage of formula (7) and for the sake of convenience write the integrand function in polar coordinates:

$$f(x_1, y_1)=\sqrt{\rho^2+2\rho\cos\varphi+1},$$

where $x_1=\rho\cos\varphi$, $y_1=\rho\sin\varphi$. Thus, we get

$$I\approx\pi\left[\frac{1}{4}\cdot 1+\frac{1}{8}\sum_{i=0}^{5}\sqrt{\frac{5}{3}+2\sqrt{\frac{2}{3}}\cos\frac{\pi}{3}i}\right]=$$

$$=\pi\left[0.25+\frac{1}{8}\left(\sqrt{3.2996}+2\sqrt{2.4832}+2\sqrt{0.8502}+\sqrt{0.0337}\right)\right]=$$

$$=3.533.$$

The exact value of the integral (10) is equal to $32/9=3.555$.

3. The Monte Carlo method (the method of statistic trials, see [4], [12]).

First method. Let it be required to evaluate the m-fold integral

$$I=\underbrace{\int\int\ldots\int}_{G} f(x_1x_2, \ldots, x_m)\,dx_1\ldots dx_m \tag{11}$$

over a domain G lying in a unit m-dimensional cube

$$0\leqslant x_i\leqslant 1 \quad (i=1, 2, \ldots, m).$$

Let us choose m sequences of random numbers distributed uniformly on the interval [0, 1] (for random numbers see [6], [10]):

$$\begin{array}{l} x_1^{(1)},\ x_2^{(1)},\ x_3^{(1)},\ \ldots,\ x_n^{(1)},\ \ldots, \\ x_1^{(2)},\ x_2^{(2)},\ x_3^{(2)},\ \ldots,\ x_n^{(2)},\ \ldots, \\ \ldots\ldots\ldots\ldots\ldots\ldots \\ x_1^{(m)},\ x_2^{(m)},\ x_3^{(m)},\ \ldots,\ x_n^{(m)}, \end{array}$$

Then $M_i(x_i^{(1)}, x_i^{(2)}, \ldots, x_i^{(m)})$ $(i = 1, 2, \ldots)$ may be considered as random points uniformly distributed in unit m-dimensional cube.

Let out of the total number N of random points n points be located in the domain G, the rest of them $(N-n)$ being found outside it. Then for a sufficiently large N we have the following approximate formula

$$I \approx \frac{V_G}{n} \sum_{i=1}^{n} f(M_i), \tag{12}$$

where V_G is understood as the m-dimensional volume of the domain of integration. If it is rather difficult to compute the volume V_G, we can put $V_G \approx \frac{n}{N}$, and for computing the approximate value of the integral get

$$I \approx \frac{1}{N} \sum_{i=1}^{n} f(M_i). \tag{13}$$

Example 3. Evaluate the following integral by the Monte Carlo method

$$I = \iint\limits_G (x^2 + y^2)\, dx\, dy, \tag{14}$$

where the domain G is defined by the following inequality (Fig. 2): $1/2 \leqslant x \leqslant 1$, $0 \leqslant y \leqslant 2x - 1$.

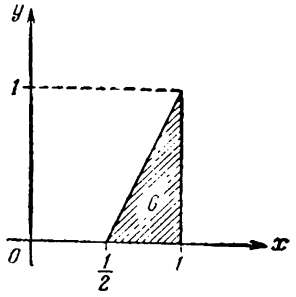

Fig. 2

Solution. The domain of integration belongs to the unit square $0 \leqslant x \leqslant 1$, $0 \leqslant y \leqslant 1$.

To evaluate the integral let us take advantage of the table of random numbers (Table 60), taking each two consecutive numbers from this table for the coordinate of the random point $M(x, y)$.

Table 60

Random Numbers Uniformly Distributed on Interval [0, 1]

0.57705	0.35483	0.11578	0.65339
0.71618	0.09393	0.93045	0.93382
0.73710	0.30304	0.93011	0.05758
0.70131	0.55186	0.42844	0.00336
0.16961	0.64003	0.52906	0.88222
0.53324	0.20514	0.09461	0.98585
0.43166	0.00188	0.99602	0.52103
0.26275	0.55709	0.69962	0.91827
0.05926	0.86977	0.31311	0.07069
0.66289	0.31303	0.27004	0.13928

Tabulate the coordinates x and y of the random points to form Table 61, rounding them to three decimal digits and choose the ones belonging to the domain of integration.

The order of filling in Table 61.

(1) From among all values of x single out those lying between $x = 0.5$ and $\bar{x} = 1$. For these values put $\varepsilon_1 = 1$, for all the rest taking $\varepsilon_1 = 0$. For instance, for $x = 0.577$ we have $\varepsilon_1 = 1$, for $x = 0.170$ we have $\varepsilon_1 = 0$.

(2) From among all values of y, corresponding to the singled out x, choose these contained between

$$\underline{y(x)} = 0, \quad \overline{y(x)} = 2x - 1.$$

For these values we put $\varepsilon_2 = 1$, for all the rest taking $\varepsilon_2 = 0$. For instance, for $y = 0.701$ we obtain $\varepsilon_2 = 0$, for $y = 0.205$ we get $\varepsilon_2 = 1$.

(3) Compute $\varepsilon = \varepsilon_1 \varepsilon_2$. Only the points for which $\varepsilon = 1$ belong to the domain of integration. For instance, for the point $M(0.640; 0.205)$ we have $\varepsilon = 1$. In our example only four points belong to the domain of integration, i.e. $N = 20$, $n = 4$.

(4) Evaluate the integrand function for the obtained points.

On filling in Table 61 compute the area of the domain of integration

$$V_G = \frac{1}{2} \cdot 1 \cdot \frac{1}{2} = \frac{1}{4}$$

and, using formula (12), find

Table 61

Evaluating the Double Integral (14) by the Monte Carlo Method

x	$\underline{x}$	$\overline{x}$	ε_1	y	$\underline{y(x)}$	$\overline{y(x)}$	ε_2	ε	$f(x, y)$
0.577	0.500	1.000	1	0.716	0	0.154	0	0	
0.737	0.500	1.000	1	0.701	0	0.474	0	0	
0.170	0.500	1.000	0	0.533				0	
0.432	0.500	1.000	0	0.263				0	
0.059	0.500	1.000	0	0.663				0	
0.355	0.500	1.000	0	0.094				0	
0.303	0.500	1.000	0	0.552				0	
0.640	0.500	1.000	1	0.205	0	0.280	1	1	0.452
0.002	0.500	1.000	0	0.557				0	
0.870	0.500	1.000	1	0.323	0	0.740	1	1	0.855
0.116	0.500	1.000	0	0.930				0	
0.930	0.500	1.000	1	0.428	0	0.860	1	1	1.048
0.529	0.500	1.000	1	0.095	0	0.058	0	0	
0.996	0.500	1.000	1	0.700	0	0.992	1	1	1.482
0.313	0.500	1.000	0	0.270				0	
0.653	0.500	1.000	1	0.934	0	0.306	0	0	
0.058	0.500	1.000	0	0.003				0	
0.882	0.500	1.000	1	0.986	0	0.764	0	0	
0.521	0.500	1.000	1	0.918	0	0.042	0	0	
0.071	0.500	1.000	0	0.139				0	
							Σ	4	3.837

$$I \approx \frac{1}{4} \cdot \frac{1}{4} \cdot (0.452 + 0.855 + 1.048 + 1.482) = \frac{1}{16} \cdot 3.837 = 0.240.$$

By way of comparison we give the exact value of the integral $I = \frac{7}{32} = 0.21875$.

The result obtained is of a comparatively low accuracy because the number of points $N = 20$ is not sufficiently large.

Second method. If the function $y = f(x_1, x_2, \ldots, x_m) \geqslant 0$, then the integral (11) may be considered as the volume of a solid in $(m+1)$-dimensional space, i.e.

$$I = \iint \ldots \int_V dx_1 dx_2 \ldots dx_m dy, \tag{15}$$

where the domain of integration V is determined by the following conditions

$$x = (x_1, x_2, \ldots, x_m) \in G, \quad 0 \leqslant y \leqslant f(x).$$

If in the domain G

$$0 \leqslant f(x) \leqslant B, \quad 0 \leqslant x_i \leqslant 1 \quad (i = 1, 2, \ldots, m),$$

then, introducing a new variable

$$\eta = \frac{1}{B} y,$$

we get

$$I = B \int\int \ldots \int_{v} dx_1 dx_2 \ldots dx_m d\eta,$$

where the domain v lies in the unit $(m+1)$-dimensional cube

$$0 \leqslant x_i \leqslant 1 \quad (i = 1, 2, \ldots, m), \quad 0 \leqslant \eta \leqslant 1.$$

Let us take on the interval [0, 1] $m+1$ uniformly distributed random sequences:

$$\begin{array}{llllll} x_1^{(1)}, & x_2^{(1)}, & x_3^{(1)}, & \ldots, & x_n^{(1)}, & \ldots, \\ x_1^{(2)}, & x_2^{(2)}, & x_3^{(2)}, & \ldots, & x_n^{(2)}, & \ldots, \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ x_1^{(m)}, & x_2^{(m)}, & x_3^{(m)}, & \ldots, & x_n^{(m)}, & \ldots, \\ \eta_1, & \eta_2, & \eta_3, & \ldots, & \eta_n, & \ldots \end{array}$$

Form the corresponding sequences of random points

$$M_i(x_i^{(1)}, \quad x_i^{(2)}, \quad \ldots, \quad x_i^{(m)}, \quad \eta_i) \quad (i = 1, 2, \ldots).$$

Let out of the total number N of random points n points belong to the volume v, then the following approximate formula is valid:

$$I \approx B \cdot \frac{n}{N}. \tag{16}$$

Example 4. Compute the approximate value of the volume bounded by the surfaces (Fig. 3)

$$z = 2 + \sqrt{(0.5)^2 - (x-0.5)^2 - (y-0.5)^2},$$
$$(x-0.5)^2 + (y-0.5)^2 = (0.5)^2, \quad z = 0.$$

Solution. The required volume is numerically equal to the value of the integral

$$I = \iiint_V dx\, dy\, dz. \tag{17}$$

Since in the domain V

$$0 \leqslant z \leqslant 2.5,$$

introduce the new variable

$$\eta = \frac{z}{2.5},$$

which transform the integral (17) into the integral

$$I = 2.5 \iiint_V dx\, dy\, d\eta, \tag{18}$$

where v is the domain bounded by the surfaces

$$(x-0.5)^2+(y-0.5)^2=(0.5)^2,$$

$$\eta=0.8+0.4\sqrt{(0.5)^2-(x-0.5)^2-(y-0.5)^2}, \quad \eta=0,$$

i.e. v belongs to the unit cube

$$0\leqslant x\leqslant 1, \quad 0\leqslant y\leqslant 1, \quad 0\leqslant \eta\leqslant 1.$$

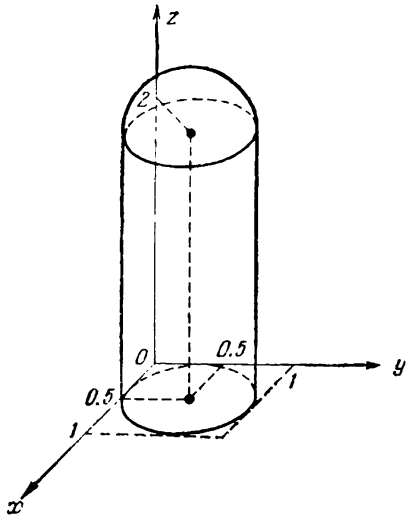

Fig. 3

Now take on the interval [0, 1] three uniformly distributed sequences of random numbers and enter them as the coordinates x, y, η in Table 62 (for instance, for the coordinates x we take the numbers of the first two columns of Table 60, and for y—the numbers of the last two columns). Then check which of these points belong to the domain v.

The order of filling in Table 62.

(1) Single out the points for which $\eta\leqslant 0.8$, and put for them $\varepsilon_2=1$.

(2) Out of these points to the domain v belong those for which the following inequality is fulfilled

$$(x-0.5)^2+(y-0.5)^2\leqslant(0.5)^2.$$

For these points $\varepsilon_1=1$, for all the rest $\varepsilon_1=0$.

(3) Compute $\varepsilon=\varepsilon_1\varepsilon_2$. The points for which $\varepsilon=1$ belong to the domain v.

Table 62

Evaluating the Integral (18) by the Monte Carlo Method

i	x	y	$\lvert x-0.5\rvert$	$\lvert y-0.5\rvert$	$(x-0.5)^2+(y-0.5)^2$	ε_1	η	$\lvert\eta-0.8\rvert$	$6.25(\eta-0.8)^2$	ε_2	ε
1	0.577	0.116	0.077	0.384	0.147	1	0.667			1	1
2	0.716	0.930	0.216	0.430	0.232		0.993	0.193	0.231		0
3	0.737	0.930	0.237	0.430	0.241	1	0.242			1	1
4	0.701	0.428	0.201	0.072	0.045		0.940	0.140	0.122		1
5	0.170	0.529	0.330	0.029	0.110	1	0.610			1	1
6	0.533	0.095	0.033	0.405	0.165	1	0.131			1	1
7	0.432	0.996	0.068	0.496	0.251	0	0 352			1	0
8	0.263	0.699	0.237	0.199	0.096	1	0.645			1	1
9	0.059	0.313	0.441	0.187	0.229	1	0.646			1	1
10	0.663	0.270	0.163	0.230	0.080	1	0.680			1	1
11	0.355	0.653	0.145	0.153	0.046	1	0.577			1	1
12	0.094	0.934	0.406	0.434	0.353	0	0.716			1	0
13	0.303	0.058	0.197	0.442	0.234	1	0.737			1	1
14	0.552	0.003	0.052	0.497	0.250	1	0.701			1	1
15	0.640	0.882	0.140	0.382	0.165	1	0.169			1	1
16	0.205	0.986	0.295	0.486	0.323	0	0.533			1	0
17	0.002	0.521	0.498	0.021	0.248	1	0.432			1	1
18	0.557	0.918	0.057	0.418	0.178	1	0.263			1	1
19	0.870	0.071	0.370	0.429	0.318	0	0.059			1	0
20	0.313	0.139	0.187	0.361	0.185	1	0.663			1	1
										$n=15$	

(4) Out of the points for which $0.8 \leqslant \eta < 1$ to the domain v belong the points whose coordinates satisfy the inequality

$$(x-0.5)^2+(y-0.5)^2+6.25\,(\eta-0.8)^2 \leqslant (0.5)^2.$$

For these points $\varepsilon = 1$.

In our example the total number of the points $N = 20$, and the number of the points belonging to the domain v is equal to 15. By formula (16) we get

$$I \approx 2.5\,\frac{n}{N} = 2.5\,\frac{15}{20} = 1.875.$$

The exact value of the volume v is equal to 1.833.

Note 1. The error of formula (16) is inversely proportional to the square root of the number of the tests, i. e.

$$R = 0\left(\frac{1}{\sqrt{N}}\right).$$

This means that to ensure a high accuracy the number of points N must be very large. But since the approximate formulas (13), (16) are independent of the

order of an integral, the Monte Carlo method turns out to be suitable for computing integrals of high order.

Note 2. Application of the Monte Carlo method is embarrassed by generating sequences of random (or pseudo-random) numbers. At present some new methods of obtaining such numbers on a computer are developed (see, for instance, [49]). Tables of random numbers may be also used (see [3], [15]).

Problems

Evaluate the integrals by the method of iterated integrating, using various quadrature formulas. In Problems 1 to 3 compare the obtained value with the exact value of the integral.

1. $\int_0^2 dy \int_0^1 (x^2+2y)\,dx$ by the trapezoidal formula for $n_x=4$, $n_y=8$.

2. $\int_3^4 dx \int_1^2 \frac{dy}{(x+y)^2}$ by Simpson's formula for $n_x=n_y=4$.

3. $\int_0^1 dx \int_0^1 \frac{x^2\,dy}{1+y^2}$ by Gauss' formula for $n_x=n_y=4$.

4. $\int_0^1 \int_1^2 \frac{\sin(x^2+y^2)}{1+ax+by}\,dx\,dy$, $a=0.5+0.1\cdot n$, $n=0, 1, 2, 3, 4, 5$, $b=0.5+0.1\cdot k$, $k=0, 1, 2, 3, 4, 5$.

5. $\int_0^2 \int_{0.5}^{1.5} \frac{e^{-a(x+y)}}{1+b(x+y)}\,dx\,dy$, $a=0.2\cdot n$, $n=0, 1, 2, 3, 4, 5$, $b=0.1\cdot k$, $k=0, 1, 2, 3, 4, 5$.

6. Evaluate the integrals in Problems 1 to 3 by formula (9).

Evaluate the following integrals using the Lyusternik-Ditkin formulas (7) and (8).

7. $\iint_G \sqrt{1+x^2+y^2}\,dx\,dy$, where G is the domain $x^2+y^2\leqslant 2x$.

8. $\iint_G \frac{x^2y^2\,dx\,dy}{x^2+y^2}$, where G is the domain $x^2+y^2\leqslant 4$.

9. $\iint_G \frac{(1+x)\,dx\,dy}{\sqrt{1+x^2+y^2}}$, where G is a regular hexagon inscribed in the unit circle.

10. $\iint_G \frac{x^2}{\sqrt{2-\frac{x^2}{4}-\frac{y^2}{9}}}\,dx\,dy$, where G is the domain $\frac{x^2}{4}+\frac{y^2}{9}\leqslant 1$.

Using the Monte Carlo method, compute the following integrals:

11. $\iint\limits_G \sqrt{x^2-y^2}\,dx\,dy$, where G is a triangle with the vertices $O(0, 0)$, $A(1, 0)$, $B(1, 1)$.

12. $\iint\limits_G e^{x/y}\,dx\,dy$, where G is a curvilinear triangle bounded by the parabola $y^2=x$ and straight lines $x=0$, $y=1$.

13. $\iiint\limits_G \frac{dx\,dy\,dz}{(x+y+z+1)^3}$, where G is the domain of integration bounded by the coordinate planes and the plane $x+y+z=1$.

14. Compute the volume of a solid bounded by the plane $z=0$, the surface $z=e^{-(x^2+y^2)}$, and the cylinder $x^2+y^2=1$.

8

APPROXIMATE SOLUTION OF ORDINARY DIFFERENTIAL EQUATIONS

8.1 CAUCHY'S PROBLEM. GENERAL REMARKS

Cauchy's problem for the differential equation of the nth order

$$y^{(n)} = f(x,\ y,\ y',\ \ldots,\ y^{(n-1)}) \tag{1}$$

consists in finding the function $y = y(x)$ satisfying this equation and the initial conditions

$$y(x_0) = y_0, \quad y'(x_0) = y_0', \ \ldots,\ y^{(n-1)}(x_0) = y_0^{(n-1)}, \tag{2}$$

where $x_0,\ y_0,\ y_0',\ \ldots,\ y_0^{(n-1)}$ are the given numbers.

Cauchy's problem for a system of differential equations

$$\left.\begin{aligned} \frac{dy_1}{dx} &= f_1(x,\ y_1,\ y_2,\ \ldots,\ y_n), \\ \frac{dy_2}{dx} &= f_2(x,\ y_1,\ y_2,\ \ldots,\ y_n), \\ &\ldots\ldots\ldots\ldots \\ \frac{dy_n}{dx} &= f_n(x,\ y_1,\ y_2,\ \ldots,\ y_n) \end{aligned}\right\} \tag{3}$$

consists in finding the functions $y_1,\ y_2,\ \ldots,\ y_n$ satisfying this system and the initial conditions

$$y_1(x_0) = y_{10}, \quad y_2(x_0) = y_{20}, \quad \ldots,\ y_n(x_0) = y_{n0}. \tag{4}$$

A system containing higher-order derivatives and solved with respect to senior derivatives of the required functions by introducing new unknown functions can be reduced to the form (3). In particular, the differential equation of the nth order

$$y^{(n)} = f(x,\ y,\ y',\ \ldots,\ y^{(n-1)})$$

is reduced to the form (3) with the aid of the substitution

$$y_1 = y', \quad y_2 = y'', \quad \ldots, \quad y_{n-1} = y^{(n-1)},$$

which gives the following system:

$$\frac{dy}{dx} = y_1,$$

$$\frac{dy_1}{dx} = y_2,$$

. . . . . .

$$\frac{dy_{n-2}}{dx} = y_{n-1},$$

$$\frac{dy_{n-1}}{dx} = f(x, y_1, y_2, \ldots, y_{n-1}).$$

If the general solution of equation (1) or system (3) is successfully found, then Cauchy's problem is reduced to finding the values of arbitrary constants. But it is rather difficult to find the exact solution of Cauchy's problem and it is successfully found only in rare cases; more often we have to solve Cauchy's problem using approximate methods.

According to the form in which the solution is represented, the approximate methods are divided into two groups:

1. *Analytic methods* yielding the approximate solution of a differential equation in the form of an analytic expression.

2. *Numerical methods* presenting the approximate solution in the form of a table. From this section on it is supposed that for the considered equations the conditions of existence and uniqueness of solution are fulfilled.

8.2 INTEGRATING DIFFERENTIAL EQUATIONS WITH THE AID OF SERIES

1. Method of successive differentiation. Consider equation (1) with initial conditions (2) of Sec. 8.1. Suppose the required partial solution $y = y(x)$ can be expanded in a Taylor's series in powers of the difference $x - x_0$:

$$y(x) = y(x_0) + \frac{y'(x_0)}{1!}(x-x_0) + \frac{y''(x_0)}{2!}(x-x_0)^2 + \ldots$$

$$\ldots + \frac{y^{(n)}(x_0)}{n!}(x-x_0)^n + \ldots . \qquad (1)$$

The initial conditions (2) of Sec. 8.1 give us directly the values $y^{(k)}(x_0)$ for $k = 0, 1, 2, \ldots, n-1$. The value $y^{(n)}(x_0)$ is found from the equation (1) of Sec. 8.1, substituting $x = x_0$ and using the

initial conditions (2) of the same Section:

$$y^{(n)}(x_0) = f(x_0, y_0, y_0', \ldots, y_0^{(n-1)}). \quad (2)$$

The values $y^{(n+1)}(x_0)$, $y^{(n+2)}(x_0)$, ... are determined successively by differentiating the equation (1) of Sec. 8.1 and substituting $x = x_0$, $y^{(k)}(x_0) = y_{0k}$ $(k = 0, 1, 2, \ldots)$.

It is proved that if the right-hand member of equation (1) in the neighbourhood of the point $(x_0, y_0, y_0', \ldots, y_0^{(n-1)})$ is an analytic function of its arguments, then for the values of x sufficiently close to x_0 there exists a unique solution of Cauchy's problem (1), (2) of Sec. 8.1, which is expanded in Taylor's series (1). Then the partial sum of this series will be the approximate solution of the considered problem.

Analogously, this method is applied for solving systems of ordinary differential equations.

Example 1. Find the first seven terms of the expansion in a power series of the solution $y = y(x)$ of the equation

$$y'' + 0.1(y')^2 + (1 + 0.1x)y = 0,$$

with the initial conditions $y(0) = 1$, $y'(0) = 2$.

Solution. We look for a solution of the equation in the form of the series

$$y(x) = y(0) + \frac{y'(0)}{1!}x + \frac{y''(0)}{2!}x^2 + \ldots + \frac{y^{(n)}(0)}{n!}x^n + \ldots.$$

Directly from the initial conditions we have $y(0) = 1$, $y'(0) = 2$. To determine $y''(0)$ solve the given equation with respect to y'':

$$y'' = -0.1(y')^2 - (1 + 0.1x)y. \quad (3)$$

Using the initial conditions, we get

$$y''(0) = -0.1 \cdot 4 - 1 \cdot 1 = -1.4.$$

Now differentiate successively both members of equation (3) with respect to x:

$$\begin{aligned}
y''' &= -0.2y'y'' - 0.1(xy' + y) - y', \\
y^{(4)} &= -0.2(y'y''' + y''^2) - 0.1(xy'' + 2y') - y'', \\
y^{(5)} &= -0.2(y'y^{(4)} + 3y''y''') - 0.1(xy''' + 3y'') - y''', \\
y^{(6)} &= -0.2(y'y^{(5)} + 4y''y^{(4)} + 3y'''^2) - 0.1(xy^{(4)} + 4y''') - y^{(4)}. \quad (4)
\end{aligned}$$

Substituting the initial conditions and the value of $y''(0)$, we find

$$y'''(0) = -1.54, \quad y^{(4)}(0) = 1.224, \quad y^{(5)}(0) = 0.1768,$$
$$y^{(6)}(0) = -0.7308.$$

Thus, the required approximate solution is written in the form

$$y(x) \approx 1 + 2x - 0.7x^2 - 0.2567x^3 + 0.051x^4 + 0.00147x^5 - 0.00101x^6.$$

Example 2. Find the first four terms of the expansion in a power series of the solution $y = y(x)$, $z = z(x)$ of the system

$$\left.\begin{aligned} y'(x) &= y\cos x - z\sin x, \\ z'(x) &= y\sin x + z\cos x \end{aligned}\right\} \tag{4}$$

with the initial conditions $y(0) = 1$, $z(0) = 0$.

Solution. We look for the functions $y(x)$ and $z(x)$ in the form of power series

$$y(x) = y(0) + \frac{y'(0)}{1!}x + \frac{y''(0)}{2!}x^2 + \ldots + \frac{y^{(k)}(0)}{k!}x^k + \ldots, \tag{5}$$

$$z(x) = z(0) + \frac{z'(0)}{1!}x + \frac{z''(0)}{2!}x^2 + \ldots + \frac{z^{(k)}(0)}{k!}x^k + \ldots. \tag{6}$$

Directly from the initial conditions we have $y(0) = 1$, $z(0) = 0$. Putting $x = 0$ in system (4) and taking into account the initial conditions, we get

$$y'(0) = 1, \quad z'(0) = 0.$$

Differentiate system (4) with respect to x:

$$\left.\begin{aligned} y''(x) &= -(y + z')\sin x - (z - y')\cos x, \\ z''(x) &= -(z - y')\sin x + (y + z')\cos x. \end{aligned}\right\} \tag{7}$$

Hence we find

$$y''(0) = 1, \quad z''(0) = 1.$$

Differentiate system (7):

$$\left.\begin{aligned} y'''(x) &= (z - 2y' - z'')\sin x - (y + 2z' - y'')\cos x, \\ z'''(x) &= -(y + 2z' - y'')\sin x - (z - 2y' - z'')\cos x. \end{aligned}\right\}$$

We get $y'''(0) = 0$, $z'''(0) = -3$. Substituting the found values of the derivatives into series (5), (6) we can write

$$y(x) \approx 1 + x + \frac{1}{2}x^2, \quad z(x) \approx \frac{1}{2}x^2 + \frac{1}{2}x^3.$$

Note. For a successful application of some numerical methods of integrating differential equations it is required to determine the values of the sought-for functions at several points. These values can be computed with the aid of power series. Thus, the method of expanding the solutions in power series can be used as an element of more effective numerical methods of approximate integration of differential equations (the methods of Adams, Miln, etc.). Given below is an example of forming a table of solutions with a certain spacing.

If the equation contains a singularity, say, an indeterminate form of the type $\frac{0}{0}$, then a numerical solution is impossible. Then the use of power series makes it possible to "move aside" from such a singularity (see problems 11 to 13).

Example 3. For the function $y = y(x)$, satisfying the equation

$$y'' + xy' + y = 0 \tag{8}$$

and the initial conditions $y(0) = 0$, $y'(0) = 1$, form a table of va-

lues on the interval [0; 0.2] with the spacing 0.05. The solution should be obtained to an accuracy not exceeding $0.5 \cdot 10^{-4}$.

Solution. Rewrite the function $y = y(x)$ in the form of a series

$$y(x) = y(0) + \frac{y'(0)}{1!}x + \frac{y''(0)}{2!}x^2 + \ldots + \frac{y^{(n)}(0)}{n!}x^n + \ldots,$$

where $y(0) = 0$, $y'(0) = 1$.

From equation (8) we get

$$y'' = -xy' - y.$$

Hence $y''(0) = -y(0) = 0$.

Differentiating successively equation (8), we have

$$y'''(x) = -xy'' - 2y',$$
$$y^{(4)}(x) = -xy''' - 3y'',$$
$$y^{(5)}(x) = -xy^{(4)} - 4y''',$$
$$\ldots\ldots\ldots\ldots$$

and in general $y^{(n+1)}(x) = -xy^{(n)} - ny^{(n-1)}$. From these equalities we get

$$y'''(0) = -2, \quad y^{(4)}(0) = 0, \quad y^{(5)}(0) = 8.$$

And in general

$$y^{(2n)}(0) = 0, \quad (y^{(2n+1)}(0) = -2ny^{(2n-1)}(0) = (-1)^n 2^n n!.$$

Hence

$$y(x) = x - \frac{x^3}{3} + \frac{x^5}{15} - \ldots + (-1)^n \frac{2^n n!\, x^{2n+1}}{(2n+1)!} + \ldots.$$

This is an alternating series, and its terms monotonically decrease by the absolute value on the interval [0; 0.2]. Since the absolute value of the remainder term of such a series is less than that of the first rejected term, the approximate formula

$$y(x) \approx x - \frac{x^3}{3}$$

will yield the values of the required function on the interval [0, 0.2] with an error less than

$$\frac{x^5}{15} \leqslant \frac{(0.2)^5}{15} \approx 0.00002.$$

Using the obtained formula, form Table 63.

Table 63

Solving Equation (8)

x	0	0.05	0.10	0.15	0.20
$y(x)$	0	0.0500	0.0997	0.1489	0.1973

2. Method of indefinite coefficients. This method is recommended to be used for solving linear differential equations (with variable coefficients). Let us explain this method, taking as an example

the second-order equation

$$y'' + p(x)\,y' + q(x)\,y = r(x) \tag{9}$$

with the initial conditions $y(0) = y_0$, $y'(0) = y_0'$. Let us assume that each coefficient of equation (9) can be expanded in a series in powers of x:

$$p(x) = \sum_{n=0}^{\infty} p_n x^n, \qquad q(x) = \sum_{n=0}^{\infty} q_n x^n, \quad r(x) = \sum_{n=0}^{\infty} r_n x^n.$$

We shall look for the solution of the given equation in the form of the series

$$y(x) = \sum_{n=0}^{\infty} c_n x^n, \tag{10}$$

where c_n are the coefficients to be determined.

Differentiate both members of equality (10) twice with respect to x:

$$y'(x) = \sum_{n=1}^{\infty} n c_n x^{n-1}, \qquad y''(x) = \sum_{n=2}^{\infty} n(n-1)\,c_n x^{n-2}.$$

Substituting the obtained series for y, y', y'', p, q, r into equation (9), we get

$$\sum_{n=2}^{\infty} n(n-1)\,c_n x^n + \sum_{n=0}^{\infty} p_n x^n \cdot \sum_{n=1}^{\infty} n c_n x^{n-1} + \sum_{n=0}^{\infty} q_n x^n \cdot \sum_{n=0}^{\infty} c_n x^n = \sum_{n=0}^{\infty} r_n x^n. \tag{11}$$

Multiplying the series and equating the coefficients of equal powers of x in the left- and right-hand members of identity (11), we get the system

$$\left.\begin{array}{l|l} x^0 & 2c_2 + c_1 p_0 + c_0 q_0 = r_0, \\ x^1 & 3\cdot 2c_3 + 2c_2 p_0 + c_1 p_1 + c_1 q_0 + c_0 q_1 = r_1, \\ x^2 & 4\cdot 3c_4 + 3c_3 p_0 + 2c_2 p_1 + c_1 p_2 + c_2 q_0 + c_1 q_1 + c_0 q_2 = r_2, \\ \ldots & \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ x^n & (n+2)(n+1)\,c_{n+2} + L(c_{n+1}, c_n, \ldots, c_1, c_0) = q_n, \end{array}\right\} \tag{12}$$

where $L(c_{n+1}, c_n, \ldots, c_1, c_0)$ denotes the linear function of the arguments $c_0, c_1, \ldots, c_n, c_{n+1}$.

Each equation of the system (12) contains by one unknown more as compared with the previous equation. The coefficients c_0, c_1 are determined from the initial conditions, all the remaining ones being determined successively from the system (12). It is proved that if the series $\sum_{n=0}^{\infty} p_n x^n$, $\sum_{n=0}^{\infty} q_n x^n$, $\sum_{n=0}^{\infty} r_n x^n$ converge for $|x| < R$, then the power series obtained converge in the same domain and is the solution of equation (9).

Note. If the initial conditions are given for $x = x_0$, then it is advisable to make the substitution $x - x_0 = t$, thus reducing the problem to the above considered.

Example 4. Find the solution of the equation

$$y'' - xy' + y = 1 - \cos x, \tag{13}$$

satisfying to the initial conditions $y(0) = 0$, $y'(0) = 1$.

Solution. Expand the coefficients of the given equation in power series

$$p(x) = -x, \quad q(x) = 1, \quad r(x) = 1 - \cos x = \frac{x^2}{2!} - \frac{x^4}{4!} + \frac{x^6}{6!} - \ldots.$$

We shall look for the solution of equation (14) in the form of the series

$$y = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + \ldots + c_n x^n + \ldots.$$

Then

$$y' = c_1 + 2c_2 x + 3c_3 x^2 + 4c_4 x^3 + \ldots + nc_n x^{n-1} + \ldots,$$
$$y'' = 2c_2 + 6c_3 x + 12c_4 x^2 + \ldots + n(n-1)c_n x^{n-2} + \ldots.$$

Substituting the obtained series in equation (13) and equating the coefficients of equal powers of x, we get the system for determining the coefficients c_i:

$$\begin{array}{l|l}
x^0 & c_0 + 2c_2 = 0, \\
x^1 & 6c_3 = 0, \\
x^2 & -c_2 + 12c_4 = \frac{1}{2}, \\
x^3 & -2c_3 + 20c_5 = 0, \\
x^4 & -3c_4 + 30c_6 = -\frac{1}{24}, \\
x^5 & -4c_5 + 42c_7 = 0, \\
x^6 & -5c_6 + 56c_8 = \frac{1}{720}.
\end{array}$$

From the initial conditions we find $c_0 = 0$, $c_1 = 1$.

It is easy to note that $c_{2n+1} = 0$ $(n = 1, 2, \ldots)$. Then,

$$c_2 = 0, \quad c_4 = \frac{1}{24}, \quad c_6 = \frac{1}{360}, \quad c_8 = \frac{11}{40\,320}.$$

Thus, we get an approximate solution of the problem in the form

$$y(x) \approx x + \frac{x^4}{24} + \frac{x^6}{360} + \frac{11x^8}{40\,320}.$$

Note. Sometimes, when solving a differential equation by the method of indefinite coefficients, we succeed in finding an expression for the coefficients of a series in the general form. This is illustrated by the following example.

Example 5. Find the solution of the equation

$$y'' + xy' + 2y = 12, \tag{14}$$

satisfying the initial conditions $y(0) = 5$, $y'(0) = 2$, in the form of a series in powers of x.

Solution. We shall look for a solution in the form of the serie

$$y(x) = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + \ldots + c_n x^n + \ldots.$$

Then

$$y'(x) = c_1 + 2c_2 x + 3c_3 x^2 + \ldots + nc_n x^{n-1} + \ldots,$$
$$y''(x) = 2c_2 + 2 \cdot 3c_3 x + \ldots + n(n-1)c_n x^{n-2} + \ldots.$$

Substituting the obtained series in equation (14), and equating the coefficients of the equal powers of x in both members of the equation, we get the system

$$\left.\begin{array}{l|l} x^0 & 2c_2 + 2c_0 = 12, \\ x^1 & 3 \cdot 2c_3 + 3c_1 = 0, \\ x^2 & 4 \cdot 3c_4 + 4c_2 = 0, \\ x^3 & 5 \cdot 4c_5 + 5c_3 = 0, \\ \ldots & \ldots\ldots\ldots\ldots\ldots\ldots \\ x^n & (n+1)(n+2)c_{n+2} + (n+2)c_n = 0. \end{array}\right\} \tag{15}$$

From the initial conditions we have $c_0 = 5$, $c_1 = 2$.

Solving system (15), we successively find

$$c_2 = 6 - c_0 = 1, \qquad c_5 = -\frac{c_3}{4},$$
$$c_3 = -\frac{c_1}{2} = -1, \qquad \ldots\ldots\ldots$$
$$c_4 = -\frac{c_2}{3}, \qquad c_{n+2} = -\frac{c_n}{n-1}.$$

Write out separately the coefficients with odd and even subscripts:

$$c_3 = -\frac{c_1}{2}, \qquad c_4 = -\frac{c_2}{3},$$
$$c_5 = -\frac{c_3}{4}, \qquad c_6 = -\frac{c_4}{5},$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$c_{2k+1} = -\frac{c_{2k-1}}{2k}, \qquad c_{2k} = -\frac{c_{2k-2}}{2k-1},$$

wherefrom we get

$$c_{2k+1} = \frac{(-1)^k c_1}{2 \cdot 4 \ldots 2k}, \qquad c_{2k} = \frac{(-1)^{k-1} c_2}{3 \cdot 5 \cdot 7 \ldots (2k-1)}.$$

Substitute the values of c_1 and c_2; then

$$c_{2k+1} = \frac{(-1)^k \cdot 2}{(2k)!!}, \qquad c_{2k} = \frac{(-1)^{k-1}}{(2k-1)!!} \qquad (k = 1, 2, 3, \ldots).$$

Write two series: one for even, the other for odd powers of x:

$$\sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{(2k-1)!!} x^{2k}, \quad 2\sum_{k=1}^{\infty} \frac{(-1)^k}{(2k)!!} x^{2k+1}.$$

Find the domain of convergence for the obtained series.

For the first series the limit of the absolute value of the ratio of the subsequent term to the previous one

$$\lim_{k\to\infty}\left|\frac{u_{k+1}}{u_k}\right| = \lim_{k\to\infty}\left|\frac{x^{2k+2}\cdot 3\cdot 5\ldots(2k-1)}{3\cdot 5\ldots(2k-1)(2k+1)x^{2k}}\right| = x^2 \lim_{k\to\infty}\frac{1}{2k+1} = 0.$$

Analogously, for the second series

$$\lim_{k\to\infty}\left|\frac{u_{k+1}}{u_k}\right| = \lim_{k\to\infty}\left|\frac{x^{2k+3}2\cdot 4\ldots 2k}{2\cdot 4\ldots 2k(2k+2)x^{2k+1}}\right| = x^2 \lim_{k\to\infty}\frac{1}{2k+2} = 0.$$

Thus, both series converge over the whole number axis, therefore the solution of equation (14) can be written in the form

$$y(x) = 5 + 2x + \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{(2k-1)!!} x^{2k} + 2\sum_{k=1}^{\infty} \frac{(-1)^k}{(2k)!!} x^{2k+1}.$$

Problems

Applying the method of successive differentiation, find the solutions of the equations and systems of equations satisfying the given initial conditions in the form of a partial sum of the series (confine yourself to four-five terms).

1. $y' = y^2 + x^2$, $y(0) = 1/2$.
2. $y' = \cos(x+y)$, $y(0) = 0$.
3. $y' = e^y + x^2$, $y(1) = 0$.
4. $y' = x \ln y$, $y(1) = 1$.
5. $y'' + y\cos x = 0$, $y(0) = a$, $y'(0) = 0$.
6. $y'' + xy' = e^{-x^2}$, $y(0) = 1$, $y'(0) = 0$.
7. $\left.\begin{aligned} y' &= xy + z, \\ z' &= y - z, \end{aligned}\right\}$ $y(0) = 0$, $z(0) = 1$.
8. $\left.\begin{aligned} y' &= x + z^2, \\ z' &= xy, \end{aligned}\right\}$ $y(0) = 1$, $z(0) = -1$.

Find the solutions of the following equations using the method of indefinite coefficients.

9. $y'' + y' + x^2 y = \frac{x}{1-x}$, $y(0) = 0$, $y'(0) = 1$.
10. $y'' - xy' - 2y = e^{-x^2}$, $y(0) = 1$, $y'(0) = 1/2$.
11. $4xy'' + 2y' + y = 0$, $y(0) = 1$, $y'(0) = -1/2$.
12. $xy'' + y' + xy = 0$, $y(0) = 1$, $y'(0) = 0$.
13. $xy'' + 2y' + xy = 0$, $y(0) = 1$, $y'(0) = 0$.

14. Form a table of values of the solution on the interval $[0; 0.15]$ with $h=0.05$, using the obtained partial sum of the series:

(a) for equation 1, (b) for system 7, (c) for system 8.

8.3 THE METHOD OF SUCCESSIVE APPROXIMATIONS

Consider Cauchy's problem for the first-order differential equation

$$y' = f(x, y) \tag{1}$$

with the initial condition

$$y(x_0) = y_0. \tag{2}$$

The method of successive approximations consists in that the solution of $y(x)$ is obtained as the limit of a sequence of the functions $y_n(x)$, which are found by the recurrence formula

$$y_n(x) = y_0 + \int_{x_0}^{x} f(x, y_{n-1}(x))\,dx. \tag{3}$$

It is proved (see [36]), that if the right-hand member of $f(x, y)$ in some closed rectangle $R\{|x-x_0| \leqslant a,\ |y-y_0| \leqslant b\}$ satisfies Lipshits' condition with respect to y:

$$|f(x, y_1) - f(x, y_2)| \leqslant N|y_1 - y_2|, \quad N = \text{const}, \tag{4}$$

then, irrespective of the choice of the initial function, the successive approximation of $y_n(x)$ converge on some interval $[x_0, x_0+h]$ to the solution of problem (1), (2).

If $f(x, y)$ is continuous in a rectangle R, then the error of the approximate solution of $y_n(x)$ on the interval $[x_0, x_0+h]$ is estimated by the inequality

$$\varepsilon_n = |y(x) - y_n(x)| \leqslant MN^n \frac{(x-x_0)^{n+1}}{(n+1)!}, \tag{5}$$

where $M = \max\limits_{(x,\ y) \in R} |f(x, y)|$, and h is determined from the condition

$$h = \min\left(a, \frac{b}{M}\right). \tag{6}$$

Any function sufficiently close to the exact solution may be taken as the initial approximation $y_0(x)$. Sometimes, it is advantageous to take as $y_0(x)$ the approximate solution of equation (1) obtained in the form of a partial sum of the power series (Example 2).

Note. The method of successive approximations can be used for solving both a system of differential equations (Example 3) and a differential equation of

order n, if the latter is written in the form of a system. We have seen that for the solution of a differential equation to be expanded in a power series it is required that the right-hand member of the equation be analytic, whereas for using the method of successive approximations the analyticity of the right-hand member is not obligatory. Therefore, the method of successive approximations is, generally speaking, more widely used: it is also used when the expansion of the solution of a differential equation in a power series is impossible. But this method, unfortunately, has its own shortcoming, which consists in that it calls for the necessity to compute more and more cumbersome integrals.

Example 1. Find three successive approximations of the solution of the equation

$$y' = x^2 + y^2 \tag{7}$$

for the initial condition $y(0) = 0$.

Solution. Taking into consideration the initial condition, let us replace equation (7) by an integral one:

$$y(x) = \int_0^x (x^2 + y^2)\, dx.$$

We take $y_0(x) \equiv 0$ as the initial approximation. Then the first approximation is found by the formula

$$y_1(x) = \int_0^x (x^2 + y_0^2(x))dx = \int_0^x x^2\, dx = \frac{x^3}{3}.$$

The second and third approximations are obtained in a similar way:

$$y_2(x) = \int_0^x (x^2 + y_1^2)\, dx = \int_0^x \left(x^2 + \frac{x^6}{9}\right) dx = \frac{x^3}{3} + \frac{x^7}{63},$$

$$y_3(x) = \int_0^x (x^2 + y_2^2)dx = \int_0^x \left(x^2 + \frac{x^6}{9} + \frac{2}{189}x^{10} + \frac{x^{14}}{3969}\right) dx =$$

$$= \frac{x^3}{3} + \frac{x^7}{63} + \frac{2x^{11}}{2079} + \frac{x^{15}}{59\,535}.$$

Estimate the error of the last approximation by formula (5). Since the function $f(x, y) = x^2 + y^2$ is defined and continuous throughout the entire plane, we may take any numbers for a and b. For the sake of definiteness let us take $a = 1$, $b = 0.5$. Then we have

$$M = \max |f(x, y)| = \max |x^2 + y^2| = 1.25,$$
$$N = \max |f'_y(x, y)| = \max |2y| = 1.$$

According to condition (6) we choose $h = 0.4$.

Thus, on the interval [0; 0.4] we get

$$|y(x)-y_3(x)| \leqslant 1.25 \cdot 1^3 \cdot \frac{x^4}{4!} = \frac{5}{96} x^4,$$

and, consequently

$$\max_{[0;\ 0.4]} |y(x)-y_3(x)| \leqslant \frac{5 \cdot (0.4)^4}{96} \approx 0.00133.$$

Note. Formula (5) often overestimates the error. In using the method of successive approximations, we take for practical purposes such an n, for which the values y_{n-1}, y_n coincide within the limits of permissible accuracy (see Example 2).

Example 2. For the equation

$$y' = x + 0.1y^2 \tag{8}$$

with the initial condition $y(0) = 1$ find the approximate solution on the interval [0; 0.2] accurate to 10^{-5}.

Solution. Choose the initial approximation $y_0(x)$ in the form

$$y_0(x) = y_0 + \frac{y'(0)}{1!} x + \frac{y''(0)}{2!} x^2.$$

To this end from equation (8) and the initial condition we determine

$$y_0 = 1,$$
$$y'(0) = 0.1y_0^2 = 0.1,$$
$$y''(0) = 1 + 0.2y_0y_0' = 1.02.$$

Thus, $y_0(x) = 1 + 0.1x + 0.51x^2$.

We find

$$f(x,\ y_0) = x + 0.1y_0^2 = 0.1 + 1.02x + 0.103x^2 + 0.0102x^3 + 0.0260x^4$$

and compute the first approximation

$$y_1(x) = 1 + \int_0^x (x + 0.1y_0^2)\,dx =$$
$$= 1 + 0.1x + 0.51x^2 + 0.034x^3 + 0.0025x^4 + 0.0052x^5.$$

Consider the difference $y_1(x) - y_0(x) = 0.034x^3 + 0.0025x^4 + 0.0052x^5$. At $x = 0.2$ it has the maximum value

$$\max_{[0;\ 0.2]} |y_1(x) - y_0(x)| = 0.00028 > 10^{-5},$$

hence, the preassigned accuracy is not yet achieved. Note that in the expression for y_1 the sum of the last two terms does not exceed 10^{-5}, therefore we may put

$$y_1(x) = 1 + 0.1x + 0.51x^2 + 0.034x^3.$$

We find

$$f(x,\ y_1) = x + 0.1y_1^2 = 0.1 + 1.02x + 0.103x^2 + 0.0170x^3 + 0.0267x^4 + \\ + 0.0017x^5 + 0.0001x^6$$

and compute the second approximation

$$y_2(x) = 1 + \int_0^x (x + 0.1y_1^2)\,dx = 1 + 0.1x + 0.51x^2 + 0.034x^3 + \\ + 0.0042x^4 + 0.0053x^5.$$

Estimate the difference $y_2(x) - y_1(x)$ on the interval $[0;\ 0.2]$:

$$|y_2(x) - y_1(x)| = 0.0042x^4 + 0.0053x^5 < 0.000008 < 10^{-5}.$$

Thus, we get

$$y(x) \approx 1 + 0.1x + 0.51x^2 + 0.034x^3 + 0.0042x^4 + 0.0053x^5.$$

Example 3. Given a system

$$\left.\begin{aligned} \frac{dy}{dx} &= x + yz, \\ \frac{dz}{dx} &= x^2 - y^2 \end{aligned}\right\} \tag{9}$$

with the initial conditions $y(0) = 1$, $z(0) = \frac{1}{2}$. Using the method of successive approximations, find the solution of this system on the interval $[0;\ 0.3]$ to within $5 \cdot 10^{-3}$.

Solution. Write system (9) in the integral form:

$$y = 1 + \int_0^x (x + yz)\,dx, \quad z = \frac{1}{2} + \int_0^x (x^2 - y^2)\,dx.$$

Using the initial values, from system (9) we find $y'(0) = 1./2$, $z'(0) = -1$, therefore, we choose

$$y_0(x) = 1 + \frac{1}{2}x, \quad z_0(x) = \frac{1}{2} - x$$

as the initial approximations.

Compute the following approximations of $y_1(x)$ and $z_1(x)$:

$$y_1(x) = 1 + \int_0^x \left(x + \frac{1}{2} - \frac{3}{4}x - \frac{1}{2}x^2\right)dx = 1 + \frac{1}{2}x + \frac{1}{8}x^2 - \frac{1}{6}x^3,$$

$$z_1(x) = \frac{1}{2} + \int_0^x \left(x^2 - 1 - x - \frac{1}{4}x^2\right)dx = \frac{1}{2} - x - \frac{1}{2}x^2 + \frac{1}{4}x^3.$$

Analogously, we get

$$y_2(x) = 1 + \frac{1}{2}x + \frac{1}{8}x^2 - \frac{5}{16}x^3 - \frac{5}{96}x^4 + \frac{11}{240}x^5 + \frac{11}{576}x^6 - \frac{1}{168}x^7,$$

$$z_2(x) = \frac{1}{2} - x - \frac{1}{2}x^2 + \frac{1}{6}x^3 + \frac{5}{96}x^4 + \frac{29}{960}x^5 + \frac{1}{144}x^6 - \frac{1}{252}x^7.$$

Estimate the differences $y_1(x) - y_2(x)$ and $z_1(x) - z_2(x)$ on the interval [0; 0.3]:

$$|y_1(x) - y_2(x)| = \left|-\frac{7}{48}x^3 - \frac{5}{96}x^4 + \frac{11}{240}x^5 + \frac{11}{576}x^6 - \frac{1}{168}x^7\right| \leqslant$$

$$\leqslant x^3\left|\frac{7}{48} + \frac{5}{96}x + \frac{1}{168}x^4\right| \leqslant 0.0043,$$

$$|z_1(x) - z_2(x)| = \left|\frac{1}{12}x^3 - \frac{5}{96}x^4 - \frac{29}{960}x^5 - \frac{1}{144}x^6 - \frac{1}{252}x^7\right| \leqslant$$

$$\leqslant x^3\left|\frac{1}{12} + \frac{1}{252}x^4\right| \leqslant 0.0024.$$

These differences are found within the limits of the specified accuracy, the terms containing x^4, x^5, x^6 being small on the interval [0; 0.3]. Consequently, with the given accuracy we may put $y \approx 1 + 0.5x + 0.125x^2 - 0.312x^3$, $z \approx 0.5 - x - 0.5x^2 + 0.167x^3$.

Problems

Find three successive approximations in solving the following differential equations and systems, putting $y_0(x) \equiv y_0$, $z_0(x) \equiv z_0$.

1. $y' = x^2 - y^2$, $y(0) = 0$. **2.** $y' = e^{-x} - y^2$, $y(0) = 1$.

3. $y' = 2x - 1 + y^2$, $y(0) = 1$. **4.** $y' = \sqrt{x} + y^2$, $y(0) = 0$.

5. $y' = xy + \sqrt{x}$, $y(0) = 0$. **6.** $y' = y\sin x + x$, $y(0) = 0$.

7. $y' = x + y\cos x$, $y(0) = 0$.

8. $\left.\begin{aligned} y' &= x + y + z, \\ z' &= y - z, \end{aligned}\right\}$ $y(0) = 1$, $z(0) = -2$.

9. $\left.\begin{aligned} y' &= xy + z, \\ z' &= y + xz, \end{aligned}\right\}$ $y(0) = 0$, $z(0) = 1$.

10. $\left.\begin{aligned} y' &= x - z^2, \\ z' &= x + y, \end{aligned}\right\}$ $y(0) = 0$, $z(0) = 1$.

11. $\left.\begin{aligned} y' &= y - z, \\ z' &= yz, \end{aligned}\right\}$ $y(0) = 0$, $z(0) = 0.5$.

12. $y' = x + y^2$, $y(0) = 0$. **13.** $y' = x + y$, $y(0) = 1$.

14. $y' = 2y - 2x^2 - 3$, $y(0) = 2$. **15.** $xy' = 2x - y$, $y(1) = 2$.

Using the method of successive approximations, find the approximate solutions of the following differential equations on the interval [0, 1] to within 10^{-3}, putting $y_0(x) \equiv y_0$.

16. $y' = 2x - 0.01y^2$, $y(0) = 1$. **17.** $y' = 4.5\sqrt{x} + 0.1y^2$, $y(0) = 0$. **18.** $y' = x + y\sqrt{x}$, $y(0) = 0$.

8.4 EULER'S METHOD

The previous sections have dealt with the analytic approximate methods of solving Cauchy's problem. The Euler method belongs to the numerical methods presenting the solution in the form of a table of approximate values of the required function $y(x)$.

Consider a differential equation

$$y' = f(x, y) \tag{1}$$

with the initial condition

$$y(x_0) = y_0. \tag{2}$$

Having chosen a sufficiently small interval h, let us construct a system of equally spaced points $x_i = x_0 + ih$ $(i = 0, 1, 2, \ldots)$.

In Euler's method approximate values of $y(x_i) \approx y_i$ are computed successively by the formula

$$y_{i+1} = y_i + hf(x_i, y_i) \quad (i = 0, 1, 2, \ldots). \tag{3}$$

Here the required integral curve $y = y(x)$, passing through the point $M_0(x_0, y_0)$, is replaced by a polygonal line $M_0M_1M_2\ldots$ with the vertices $M_i(x_i, y_i)$ $(i = 0, 1, 2, \ldots)$. Each segment M_iM_{i+1} of this line, called *Euler's polygon*, has the direction coinciding with that of the integral curve of equation (1) which passes through the point M_i.

If the right-hand member of equation (1) in some rectangle $R\{|x - x_0| \leqslant a, |y - y_0| \leqslant b\}$ satisfies the conditions

$$|f(x, y_1) - f(x, y_2)| \leqslant N|y_1 - y_2| \quad (N = \text{const}), \tag{4}$$

$$\left|\frac{df}{dx}\right| = \left|\frac{\partial f}{\partial x} + f\frac{\partial f}{\partial y}\right| \leqslant M \quad (M = \text{const}), \tag{5}$$

then we have the following error estimate:

$$|y(x_n) - y_n| \leqslant \frac{hM}{2N}[(1 + hN)^n - 1], \tag{6}$$

where $y(x_n)$ is the value of the exact solution of the equation for $x = x_n$, and y_n is the approximate value obtained for the nth pitch.

Formula (5) has only theoretical application. For practical purposes sometimes it turns out to be more convenient to use *double computation*: the calculation is repeated with the spacing $\frac{h}{2}$, and the error of the more accurate value y_n^* $\left(\text{obtained at } \frac{h}{2}\right)$ is esti-

mated approximately in the following way:

$$|y_n^* - y(x_n)| \approx |y_n^* - y_n|. \tag{7}$$

Euler's method is readily applied to systems of differential equations, as well as to differential equations of higher orders. The latter should be first reduced to a system of differential equations of the first order (Example 2).

Consider the system of two equations of the first order.

$$\left.\begin{aligned} y' &= f_1(x, y, z), \\ z' &= f_2(x, y, z) \end{aligned}\right\} \tag{8}$$

for the initial conditions $y(x_0) = y_0$, $z(x_0) = z_0$.

The approximate values $y(x_i) \approx y_i$ and $z(x_i) = z_i$ are computed successively by the formulas

$$\left.\begin{aligned} y_{i+1} &= y_i + hf_1(x_i, y_i, z_i), \\ z_{i+1} &= z_i + hf_2(x_i, y_i, z_i) \quad (i = 0, 1, 2, \ldots). \end{aligned}\right\} \tag{9}$$

Example 1. Applying Euler's method, form on the interval [0, 1] a table of values of the solution of the equation

$$y' = y - \frac{2x}{y} \tag{10}$$

with the initial condition $y(0) = 1$ for $h = 0.2$.

Solution. The results of computations are given in Table 64.

Table 64

Solving Differential Equation (10) by Euler's Method

i	x_i	y_i	Computing $f(x_i, y_i)$		Δy_i	Exact solution of $y = \sqrt{2x+1}$
			$\frac{2x_i}{y_i}$	$y_i - \frac{2x_i}{y_i}$		
0	0	1.0000	0	1.0000	0.2000	1.0000
1	0.2	1.2000	0.3333	0.8667	0.1733	1.1832
2	0.4	1.3733	0.5928	0.7805	0.1561	1.3416
3	0.6	1.5294	0.7846	0.7458	0.1492	1.4832
4	0.8	1.6786	0.9532	0.7254	0.1451	1.6124
5	1.0	1.8237				1.7320

It is filled in as follows. The initial values $x_0 = 0$, $y_0 = 1.0000$ (for $i = 0$) are entered in the first row. Using them, we compute $f(x_0, y_0) = 1$, and then $\Delta y_0 = hf(x_0, y_0) = 0.2$. Now by formula (4) for $i = 0$ we get

$$y_1 = 1 + 0.2 = 1.2.$$

The values $x_1 = 0.2$, $y_1 = 1.2000$ are entered in the second row (for $i = 1$). Using them, we can calculate $f(x_1, y_1) = 0.8667$, and then $\Delta y_1 = hf(x_1, y_1) = 0.2 \cdot 0.8667 = 0.1733$. Thus, we get

$$y_2 = y_1 + \Delta y_1 = 1.2 + 0.1733 = 1.3733.$$

For $i = 2, 3, 4, 5$ the computations are carried out analogously.

For the sake of comparison the last column of the table gives the values of the exact solution of $y = \sqrt{2x+1}$. As is seen from the table, the absolute error of y_5 is $\varepsilon = 0.0917$, i.e. the relative error constitutes 5 per cent.

Example 2. Using Euler's method, compile for the interval [1; 1.5] a table of values solving the equation

$$y'' + \frac{y'}{x} + y = 0 \tag{11}$$

with the initial conditions $y(1) = 0.77$, $y'(1) = -0.44$, for the interval $h = 0.1$.

Solution. With the aid of the substitutions $y' = z$, $y'' = z'$ replace equation (11) by the system of equations

$$y' = z, \quad z' = -\frac{z}{x} - y$$

with the initial conditions $y(1) = 0.77$, $z(1) = -0.44$. Thus, $f_1(x, y, z) = z$, $f_2(x, y, z) = -\frac{z}{x} - y$. The results of computations by formula (9) are entered in Table 65 (the computations are performed with one extra digit). The table is filled in as follows. Form the first row: $i = 0$, $x_0 = 1.0$, $y_0 = 0.77$, $z_0 = -0.44$. Then we find

Table 65

Solving Equation (11) by Euler's Method

i	x_i	y_i	Δy_i	$f_{1i} = z_i$	Δz_i	$f_{2i} = -\frac{z_i}{x_i} - y_i$
0	1.0	0.77	—0.044	—0.44	—0.033	—0.33
1	1.1	0.726	—0.047	—0.473	—0.030	—0.296
2	1.2	0.679	—0.050	—0.503	—0.026	—0.260
3	1.3	0.629	—0.053	—0.529	—0.022	—0.222
4	1.4	0.576	—0.055	—0.551		
5	1.5	0.521				

$$f_{10} = f_1(x_0, y_0, z_0) = z_0 - 0.44,$$

$$f_{20} = f_2(x_0, y_0, z_0) = -\frac{z_0}{x_0} - y_0 = -0.33.$$

Using formulas (9), we get

$$\Delta y_0 = hf_{10} = 0.1\,(-0.44) = -0.044, \quad y_1 = y_0 + \Delta y_0 = 0.726,$$
$$\Delta z_0 = hf_{20} = 0.1\,(-0.33) = -0.033, \quad z_1 = z_0 + \Delta z_0 = -0.473.$$

Thus, we can enter the second row of the table as $i = 1$, $x_1 = 1.1$, $y_1 = 0.726$, $z_1 = -0.473$. By these values we find

$$f_{11} = f_1(x_1, y_1, z_1) = z_1 = -0.473,$$
$$f_{21} = f_2(x_1, y_1, z_1) = -\frac{z_1}{x_1} - y_1 = -0.296.$$

And hence,

$$\Delta y_1 = hf_{11} = 0.1\,(-0.47) = -0.047, \quad y_2 = y_1 + \Delta y_1 = 0.679,$$
$$\Delta z_1 = hf_{21} = 0.1\,(-0.30) = -0.030, \quad z_2 = z_1 + \Delta z_1 = -0.503.$$

For $i = 2, 3, 4, 5$ the table is entered in a similar way.

Problems

Using Euler's method, compute numerically the following differential equations and systems of differential equations with the given initial conditions on the interval $[a, b]$ with the spacing $h = 0.1$ for the indicated values of the parameters.

1. $y' = \frac{1}{2}xy$, $y(0) = 1$, $a = 0$, $b = 1$.

2. $y' = x^2 + y^2$, $y(0) = 0$, $a = 0$, $b = 1$.

3. $y' = 1 + xy^2$, $y(0) = 0$, $a = 0$, $b = 1$.

4. $y' = \frac{y}{x+1} - y^2$, $y(0) = 1$, $a = 0$, $b = 1$.

5. $y' = \alpha y^2 + \frac{\beta}{x^2}$, $y(1) = 1$, $a = 1$, $b = 2$.

(a) $\alpha = -1$, $\beta = 0.05 + 0.05 \cdot k$, $k = 0, 1, 2, \ldots, 7$,

(b) $\alpha = -0.5$, $\beta = 0.1 + 0.1 \cdot k$, $k = 0, 1, 2, 3$.

6. $y' = \frac{\alpha}{x^2} - \frac{y}{x} - \beta y^2$, $y(1) = \gamma$, $a = 1$, $b = 2$.

The values of the parameters α, β, γ are given in Table 66.

7. $y' = 1 + \alpha y \sin x - \beta y^2$, $y(0) = 0$, $a = 0$, $b = 1$, $\alpha = 0.2 \cdot k$, $k = 1, 2, \ldots, 5$, $\beta = 1 + 0.25 \cdot n$, $n = 0, 1, 2, 3, 4$.

8. (a)
$$\left.\begin{aligned} y' &= -xz, \\ z' &= \frac{y}{z}, \end{aligned}\right\} \quad y(0) = 0,\ z(0) = 1,\ a = 0,\ b = 1.$$

(b)
$$\left.\begin{aligned} y' &= (z - y)x, \\ z' &= (z + y)x, \end{aligned}\right\} \quad y(0) = 1,\ z(0) = 1,\ a = 0,\ b = 1.$$

Table 66

Values of the Parameters α, β, γ

Variant No.	α	β	γ	Variant No.	α	β	γ
1	0.2	0.8	0.5	11	1.0	0.25	2.0
2	0.25	2.0	0.5	12	1.0	0.5	2.0
3	0.25	1.0	0.5	13	1.0	4.0	0.5
4	0.4	0.4	1.0	14	1.6	0.4	2.0
5	0.5	2.0	0.5	15	2.0	0.5	2.0
6	0.8	0.2	2.0	16	2.5	0.4	2.5
7	0.9	0.4	1.5	17	4.0	1.0	2.0
8	1.0	1.0	1.0	18	4.0	0.25	4.0
9	1.0	1.0	2.0	19	0.9	1.6	0.75
10	1.0	2.0	2.0	20	1.0	0.75	2.0

9. $\left.\begin{aligned} y' &= -yz + \frac{\sin x}{x}, \\ z' &= -z^2 + \frac{\alpha x}{1+x^2} \end{aligned}\right\} y(0)=0,\ z(0)=-0.4122,\ a=0,\ b=1,$

$\alpha = 2.5 + \frac{\beta}{40},\ \beta = 25, 26, \ldots, 48.$

10. $\left.\begin{aligned} y' &= z-(\alpha y+\beta z)\,y, \\ z' &= e^y-(y+\alpha z)\,y, \end{aligned}\right\} y(0)=1,\ z(0)=0,\ a=0, b=1,\ \alpha=2+$
$+0.25\cdot n,\ n=0, 1, 2, 3, 4,\ \beta=0.25\cdot k,\ k=1, 2, \ldots, 5.$

11. Intensity of radiation is proportional to the amount of remaining radioactive substance. The relevant differential equation is written in the form $y' = -ky$. Putting $k = 0.01\,\frac{1}{\text{sec}}$, $t_0 = 0, y_0 = 100$ g, determine how much substance will remain at the moment $t = 100$ sec.

Find the solution by Euler's method for $h_1 = 25$, $h_2 = 10$, $h_3 = 5$. Compare the obtained results with the exact solution $y = 100e^{-kt}$: for $t = 100$ sec $y = 36.788$ g.

12. A body with the initial mass of 200 kg is in motion under the action of a permanent force of 2000 N, losing 1 kg of its mass per second. The relevant differential equation has the form

$$\frac{dv}{dt} = \frac{2000}{200-t}.$$

Assuming that at the moment $t = 0$ the body was motionless, find its velocity v in 50 seconds.

Find the solution by Euler's method for $h_1 = 10$, $h_2 = 5$, $h_3 = 2$. Compare the results of computations with the exact solution $v = 2000 \ln \frac{200}{200-t}$: for $t = 50$ sec, $v = 575.36\,\frac{\text{m}}{\text{sec}}$.

13. If we assume that the body described in Problem 12 suffers an air resistance numerically equal to the doubled value of v, then the relevant differential equation has the form

$$\frac{dv}{dt} = \frac{2000-2v}{200-t}.$$

Find its velocity for $t = 50$ sec if at $t = 0$ the body is motionless.

Find the solution by Euler's method for $h_1 = 10$, $h_2 = 5$. The exact solution of the equation has the form $v = 10t - \frac{t^2}{40}$, hence, $v(50) = 437.5 \frac{\text{m}}{\text{sec}}$.

8.5 MODIFICATIONS OF EULER'S METHOD

Euler's first improved method (see [18]) for solving the problem (1), (2) of Sec. 8.4 consists in that first the intermediate values are computed

$$\left.\begin{aligned} x_{i+1/2} &= x_i + \frac{h}{2}, \\ y_{i+1/2} &= y_i + \frac{h}{2} f_i, \\ f_{i+1/2} &= f(x_{i+1/2}, y_{i+1/2}), \end{aligned}\right\} \tag{1}$$

and then put

$$y_{i+1} = y_i + h f_{i+1/2}. \tag{2}$$

Using *the second improved method (the Euler-Cauchy method* — see [18]), first determine the "rough approximation"

$$\tilde{y}_{i+1} = y_i + h f_i, \tag{3}$$

then compute $\tilde{f}_{i+1} = f(x_{i+1}, \tilde{y}_{i+1})$ and put approximately

$$y_{i+1} = y_i + h \frac{f_i + \tilde{f}_{i+1}}{2}. \tag{4}$$

The remainder terms of Euler's first and second improved methods have the order $O(h^3)$ for each spacing (see [26]).

The error at the point x_n can be estimated with the aid of double computation: the calculation is repeated with the spacing $\frac{h}{2}$ and the error of the more accurate value $y_n^*\left(\text{for } \frac{h}{2}\right)$ is estimated approximately in the following way

$$|y_n^* - y(x_n)| \approx \frac{1}{3} |y_n^* - y_n|,$$

where $y(x)$ is the exact solution of the differential equation.

Example 1. Making use of the first and second improved methods, integrate the equation

$$y' = y - \frac{2x}{y}$$

with the initial condition $y(0) = 1$, taking $h = 0.2$. Compare the results with those of Example 1 and with the exact solution.

Solution. The first improved method. The result of computations are presented in Table 67.

Table 67

Solving Equation (5) by Euler's First Improved Method

i	x_i	y_i	$\frac{h}{2} f_i$	$x_{i+1/2}$	$y_{i+1/2}$	$\Delta y_i = h f_{i+1/2}$
0	0	1	0.1	0.1	1.1	0.1836
1	0.2	1.1836	0.0846	0.3	1.2682	0.1590
2	0.4	1.3426	0.0747	0.5	1.4173	0.1424
3	0.6	1.4850	0.0677	0.7	1.5527	0.1302
4	0.8	1.6152	0.0625	0.9	1.6777	0.1210
5	1.0	1.7362				

The table is filled in as follows. Write in the first row $i = 0$, $x_0 = 0$, $y_0 = 1$. Compute $f_0 = f(x_0, y_0) = 1$. Then by formula (1) we obtain for $x_{1/2} = 0.1$

$$y_{1/2} = 1 + 0.1 = 1.1.$$

Then we find $f(x_{1/2}, y_{1/2}) = 0.9182$ and $\Delta y_0 = hf(x_{1/2}, y_{1/2}) = 0.2 \cdot 0.9182 = 0.1836$. Now, by formula (5.2), we have

$$y_1 = y_0 + \Delta y_0 = 1 + 0.1836 = 1.1836.$$

Using this result, write in the second row $i = 1$, $x_1 = 0.2$, $y_1 = 1.1836$ and find $\frac{h}{2} f(x_1, y_1) = 0.0846$. Then compute by formula (1) for $x_{3/2} = 0.3$

$$y_{3/2} = 1.1836 + 0.0846 = 1.2682.$$

Determine $f(x_{3/2}, y_{3/2}) = 0.7942$ and $\Delta y_1 = hf(x_{3/2}, y_{3/2}) = 0.2 \cdot 0.7942 = 0.1590$. Then by formula (2) we have

$$y_2 = y_1 + \Delta y_1 = 1.1836 + 0.1590 = 1.3426.$$

For $i = 2, 3, 4, 5$ the table is filled in analogously.

The second improved method (the Euler-Cauchy method). The results of the computations by this method are given below. This table is formed in the following way: write in the first row $i = 0$, $x_0 = 0$,

Table 68

Solving Equation (5) by the Euler-Cauchy Method

i	x_i	y_i	$\frac{h}{2} f_i$	x_{i+1}	$\tilde{y}_{i+1}$	$\frac{h}{2} \tilde{f}_{i+1}$	$\Delta y_i = \frac{h}{2}(f_i + \tilde{f}_{i+1})$
0	0	1	0.1	0.2	1.2	0.0867	0.1867
1	0.2	1.1867	0.0850	0.4	1.3566	0.0767	0.1617
2	0.4	1.3484	0.0755	0.6	1.4993	0.0699	0.1454
3	0.6	1.4938	0.0690	0.8	1.6180	0.0651	0.1341
4	0.8	1.6272	0.0645	1.0	1.7569	0.0618	0.1263
5	1.0	1.7542					

$y_0 = 1$; find $f_0 = f(x_0, y_0) = 1$; by formula (3) compute $\tilde{y}_1 = 1 + 0.2 \cdot 1 = 1.2$ and enter the values $\frac{h}{2} f_0 = 0.1$, $x_1 = 0.2$, $\tilde{y}_1 = 1.2$ in the corresponding columns of the first row. Then find $\frac{h}{2} f(x_1, \tilde{y}_1) = 0.1\left(1.2 - \frac{0.4}{1.2}\right) = 0.0867$ and by formula (4) we get

$$\Delta y_0 = \frac{h}{2}(f_0 + \tilde{f}_1) = 0.1 + 0.0867 = 0.1867,$$

$$y_1 = y_0 + \Delta y_0 = 1 + 0.1867 = 1.1867.$$

Now we proceed with the second row of the table: $i = 1$, $x_1 = 0.2$, $y_1 = 1.1867$. We find

$$f_1 = f(x_1, y_1) = 1.1867 - \frac{0.4}{1.1867} = 0.8497,$$

$$\tilde{y}_2 = 1.1867 + 0.1699 = 1.3566$$

and fill in column by column: $\frac{h}{2} f_1 = 0.0850$, $x_2 = 0.4$, $\tilde{y}_2 = 1.3566$. Using the obtained values of x_2, $\tilde{y}_2$, we compute $\frac{h}{2} f(x_2, \tilde{y}_2) = 0.0767$ and by formula (4) obtain

$$\Delta y_1 = \frac{h}{2}(f_1 + \tilde{f}_2) = 0.0850 + 0.0767 = 0.1617,$$

$$y_2 = y_1 + \Delta y_1 = 1.1867 + 0.1617 = 1.3484.$$

For $i = 2, 3, 4, 5$ the table is filled in analogously. The Euler improved methods are more accurate as compared with the ordinary Euler method, which is seen from Table 69 which presents the values of the exact solution of equation (5) and those of approximate solutions contained in Tables 64, 67, and 68.

Table 69

Comparing the Exact Solution of Equation (5) and Approximate Solutions Obtained by Euler's Method and Its Modifications

x		0.2	0.4	0.6	0.8	1.0
Exact solution		1.1832	1.3416	1.4832	1.6124	1.7320
Approximate solutions obtained by	Euler's method	1.2000	1.3733	1.5294	1.6786	1.8237
	Euler's first improved method	1.1836	1.3426	1.4850	1.6152	1.7362
	The Euler-Cauchy method	1.1867	1.3484	1.4938	1.6272	1.7542

Problems

Applying one of the Euler improved methods, find the numerical solution of each of the following differential equations with the given initial condition and spacing at the indicated point.

1. $y' = x + y^2$, $y(0) = 0$, $h = 0.03$; find $y(0.3)$.
2. $y' = 1 + x - y^2$, $y(0) = 1$, $h = 0.02$; find $y(0.1)$.
3. $y' = x^2 - y^2$, $y(0) = 0$, $h = 0.1$; find $y(1)$.
4. $y' = xy - 0.1y^2$, $y(0) = 0$, $h = 0.02$; find $y(0.2)$.
5. $y' = x^3 + y^2$, $y(0) = 0.5$, $h = 0.1$; find $y(0.5)$.
6. $y' = x + \sqrt{y}$, $y(0.5) = 0.7240$, $h = 0.1$; find $y(1.5)$.
7. $y' = 2x + \cos y$, $y(0) = 0$, $h = 0.02$; find $y(0.1)$.
8. $y' = e^x - y^2$, $y(0) = 0$, $h = 0.04$; find $y(0.4)$.
9. $y' = x \ln y - y \ln x$, $y(1) = 1$, $h = 0.1$; find $y(1.6)$.
10. $y' = 0.1\left|\sqrt[3]{y} + \ln(x + y) - 1\right|$, $y(-1) = 2$, $h = 0.2$; find $y(1)$.
11. Solve Problem 11 of Section 8.4 by Euler's first and second improved methods for $h_1 = 20$, and $h_2 = 10$.

8.6 EULER'S METHOD COMPLETE WITH AN ITERATIVE PROCESS

The Euler-Cauchy method of solving the problem (1), (2) of Sec. 8.4 can be made still more accurate by applying an iterative process (see [45]) to each value of y_i. Namely, proceeding from the rough approximation

$$y_{i+1}^{(0)} = y_i + hf(x_i, y_i), \tag{1}$$

let us form an iterative process

$$y_{i+1}^{(k)} = y_i + \frac{h}{2}\left[f(x_i, y_i) + f(x_{i+1}, y_{i+1}^{(k-1)})\right]. \quad (2)$$

Iterations are continued until the corresponding decimal digits of two subsequent approximations $y_{i+1}^{(k)}$, $y_{i+1}^{(k+1)}$ coincide. Then we put

$$y_{i+1} \approx y_{i+1}^{(k+1)}.$$

As a rule, for a sufficiently small h iterations converge rapidly. If after three-four iterations the necessary number of decimal digits do not coincide, the spacing h must be decreased.

Example 1. Using the iterative method, find (to within 10^{-4}) the value $y(0.1)$ in solving the equation

$$y' = x + y$$

with the initial condition $y(0) = 1$.

Solution. Let us take the interval $h = 0.05$. By formula (1) we compute

$$y_1^{(0)} = y_0 + hf(x_0, y_0) = 1 + 0.05 \cdot 1 = 1.05.$$

Then by formula (2) we successively have

$$y_1^{(1)} = 1 + \frac{0.05}{2}(1 + 1.10) = 1.0525,$$

$$y_1^{(2)} = 1 + \frac{0.05}{2}(1 + 1.0525) = 1.05256.$$

The required accuracy has been reached. Rounding the obtained result to four digits, we get

$$y_1 = 1.0526.$$

Applying now formula (1) for $i = 1$, we compute

$$y_2^{(0)} = y_1 + hf(x_1, y_1) = 1.0526 + 0.05 \cdot 1.1026 = 1.1077.$$

With the aid of the same iteration process (2), we get

$$y_2^{(1)} = 1.0526 + 0.025\,(1.1026 + 1.2077) = 1.11036,$$
$$y_2^{(2)} = 1.0526 + 0.025\,(1.1026 + 1.2104) = 1.11042.$$

Thus,

$$y_2 = 1.1104.$$

For the sake of comparison let us compute the exact value of $y(0.1)$ by the solution formula $y = 2e^x - x - 1$:

$$y(0.1) = 2e^{0.1} - 1.1 = 1.1103.$$

Problems

Applying Euler's method with iterative process, find (accurate to 10^{-4}) the solutions of the following equations for the given initial conditions on the interval [0, 1] with the spacing $h=0.1$.

1. $y'=1-\sin(\alpha x+y)+\frac{\beta y}{2+x}$, $y(0)=0$, $\alpha=1+0.25\cdot n$, $n=0$, 1, 2, 3, 4, $\beta=-0.3+0.2\cdot k$, $k=0$, 1, 2, 3, 4.

2. $y'=\frac{\cos y}{\alpha+x}+\beta y^2$, $y(0)=0$, $\alpha=1+0.25\cdot k$, $k=0$, 1, 2, 3, 4, $\beta=-0.5+0.2\cdot n$, $n=0$, 1, 2, 3, 4.

3. $y'=-y^2+\frac{\beta x}{1+x^2}$, $y(0)=-0.4122$, $\beta=2.5+\frac{\alpha}{40}$, $\alpha=14$, 15,... ..., 37.

8.7 THE RUNGE-KUTTA METHOD

Consider the Cauchy problem for a differential equation

$$y'=f(x, y) \tag{1}$$

with the initial condition

$$y(x_0)=y_0. \tag{2}$$

Denote the approximate value of the sought-for solution at point x_i by y_i. According to the Runge-Kutta method (see [2], [13], [18]), the approximate value y_{i+1} at the next point $x_{i+1}=x_1+h$ is computed by the formulas

$$\left.\begin{aligned} y_{i+1}&=y_i+\Delta y_i,\\ \Delta y_i&=\frac{1}{6}(K_1^{(i)}+2K_2^{(i)}+2K_3^{(i)}+K_4^{(i)}),\end{aligned}\right\} \tag{3}$$

where

$$\left.\begin{aligned} K_1^{(i)}&=hf(x_i, y_i),\\ K_2^{(i)}&=hf\left(x_i+\frac{h}{2},\ y_i+\frac{K_1^{(i)}}{2}\right),\\ K_3^{(i)}&=hf\left(x_i+\frac{h}{2},\ y_i+\frac{K_2^{(i)}}{2}\right),\\ K_4^{(i)}&=hf(x_i+h,\ y_i+K_3^{(i)}).\end{aligned}\right\} \tag{4}$$

It is advisable to arrange all the computations according to the computational scheme shown in Table 70.

The table is filled in in the following order:

(1) Write the numerical values of x_0, y_0 in the first row of the table.

(2) Compute $f(x_0, y_0)$, multiply it by h and enter the result in the table as $K_1^{(0)}$.

Table 70

Computational Scheme for the Runge-Kutta Method

i	x	y	$K=hf(x, y)$	Δy
0	x_0	y_0	$K_1^{(0)}$	$K_1^{(0)}$
	$x_0+\frac{h}{2}$	$y_0+\frac{K_1^{(0)}}{2}$	$K_2^{(0)}$	$2K_2^{(0)}$
	$x_0+\frac{h}{2}$	$y_0+\frac{K_2^{(0)}}{2}$	$K_3^{(0)}$	$2K_3^{(0)}$
	x_0+h	$y_0+K_3^{(0)}$	$K_4^{(0)}$	$K_4^{(0)}$
				Δy_0
1	x_1	y_1		

(3) Write the numerical values of $x_0+\frac{h}{2}$, $y_0+\frac{K_1^{(0)}}{2}$ in the second row.

(4) Compute $f\left(x_0+\frac{h}{2}, y_0+\frac{K_1^{(0)}}{2}\right)$, multiply it by h, and enter the result in the table as $K_2^{(0)}$.

(5) Write the numerical values of $x_0+\frac{h}{2}$, $y+\frac{K_2^{(0)}}{2}$ in the third row.

(6) Compute $f\left(x_0+\frac{h}{2}, y_0+\frac{K_2^{(0)}}{2}\right)$, multiply it by h, and enter the result as $K_3^{(0)}$.

(7) Write the numerical values of x_0+h, $y_0+K_3^{(0)}$ in the fourth row of the table.

(8) Compute $f(x_0+h, y_0+K_3^{(0)})$, multiply it by h, and enter the result as $K_4^{(0)}$.

(9) Enter the numbers $K_1^{(0)}$, $2K_2^{(0)}$, $2K_3^{(0)}$, $K_4^{(0)}$ in the column Δy.

(10) Summate the numbers forming the column Δy, divide the sum by 6, and enter the result as Δy_0.

(11) Compute $y_1=y_0+\Delta y_0$.

Then continue computing following the same order, taking (x_1, y_1) for the initial point.

The results of the computation of the right-hand member of $f(x, y)$ may be included in Table 70 (see Examples 1 and 2). But if the computations are too cumbersome, it is good practice to enter them in a separate table (see Example 3).

Note that the interval of computation (the spacing) may be changed when passing from one point to another. To check h for

the proper choice it is recommended (see [18]) to compute the fraction

$$\theta = \left| \frac{K_2^{(i)} - K_3^{(i)}}{K_1^{(i)} - K_2^{(i)}} \right|. \tag{5}$$

The quantity θ should not exceed several hundredths, otherwise h should be reduced. The order of accuracy of the Runge-Kutta method is h^4 over the entire interval $[x_0, X]$. The error estimate with this method is rather difficult. The error can be roughly estimated with the aid of double computation by the formula

$$|y_n^* - y(x_n)| \approx \frac{|y_n^* - y_n|}{15},$$

where $y(x_n)$ is the value of the exact solution of equation (1) for point x_n, and y_n^*, y_n are the approximate values obtained for $\frac{h}{2}$ and h.

Example 1. Using the Runge-Kutta method, find the solution of the equation

$$y' = 0.25y^2 + x^2 \tag{6}$$

with the initial condition $y(0) = -1$ on the interval $[0; 0.5]$, taking $h = 0.1$.

Solution. The results of the computations are given in Table 71, which is formed in the following succession. For $i = 0$:

(1) Write down in the first row: $x_0 = 0$, $y_0 = -1$.

(2) Compute $f(x_0, y_0) = 0.25$, then $K_1^{(0)} = 0.1 \cdot 0.25 = 0.025$.

(3) Write down in the second row: $x_0 = \frac{h}{2} = 0.05$, $y_0 + \frac{K_1^{(0)}}{2} = -0.98750$.

(4) Compute $f\left(x_0 + \frac{h}{2}, y_0 + \frac{K_1^{(0)}}{2}\right) = 0.24629$; then $K_2^{(0)} = 0.024629$.

(5) Write down in the third row: $x_0 + \frac{h}{2} = 0.05$, $y_0 + \frac{K_2^{(0)}}{2} = -0.98769$.

(6) Compute $f\left(x_0 + \frac{h}{2}, y_0 + \frac{K_2^{(0)}}{2}\right) = 0.25 \cdot (0.98769)^2 + (0.05)^2 = 0.24638$; then $K_3^{(0)} = 0.024638$.

(7) Write down in the fourth row: $x_0 + h = 0.1$, $y_0 + K_3^{(0)} = -0.97536$.

(8) Compute $f(x_0 + h, y_0 + K_3^{(0)}) = 0.25 \cdot (0.97536)^2 + (0.1)^2 = 0.24783$; then $K_4^{(0)} = 0.024783$.

(9) Enter the numbers $K_1^{(0)}$, $2K_2^{(0)}$, $2K_3^{(0)}$, $K_4^{(0)}$ to form the column Δy.

(10) Compute $\Delta y_0 = \frac{1}{6} \cdot 0.148317 = 0.02472$.

(11) We get $y_1 = y_0 + \Delta y_0 = -0.97528$.

Table 71

Solving Equation (6) by the Runge-Kutta Method

i	x	y	$0.25y$	$K=hf(x, y)$	Δy	$\theta=\left\|\frac{K_2-K_3}{K_1-K_2}\right\|$
0	0	—1	—0.25	0.025	0.025	
	0.05	—0.98750	—0.24688	0.024629	0.049258	0.024
	0.05	—0.98769	—0.24692	0.024638	0.049276	
	0.1	—0.97536	—0.24384	0.024783	0.024783	
					0.02472	
1	0.1	—0.97528	—0.24382	0.024779	0.024779	
	0.15	—0.96289	—0.24072	0.025429	0.050858	0.025
	0.15	—0.96257	—0.24064	0.025413	0.050826	
	0.2	—0.94987	—0.23747	0.026557	0.026557	
					0.02550	
2	0.2	—0.94978	—0.23745	0.026553	0.026553	
	0.25	—0.93650	—0.23413	0.028176	0.056352	0.023
	0.25	—0.93569	—0.23392	0.028138	0.056276	
	0.3	—0.92164	—0.23041	0.030236	0.030236	
					0.02824	
3	0.3	—0.92154	—0.23039	0.030231	0.030231	
	0.35	—0.90642	—0.22661	0.032780	0.065580	0.023
	0.35	—0.90514	—0.22629	0.032732	0.065464	
	0.4	—0.88881	—0.22220	0.035743	0.035749	
					0.03284	
4	0.4	—0.88870	—0.22218	0.035745	0.035745	
	0.45	—0.87083	—0.21771	0.039209	0.078418	0.022
	0.45	—0.86910	—0.21728	0.039134	0.078268	
	0.5	—0.84957	—0.21239	0.04307	0.043044	
					0.03925	
5	0.5	—0.84945				

Enter the values $x_1=0.1$, $y_1=-0.97528$ in the row marked "$i=1$", and carry out the computations by formulas (3), (4) once again.

The last column of the table provides the values of θ obtained by formula (5).

Example 2. Using the Runge-Kutta method find to within $5\cdot10^{-6}$

the solution of the differential equation

$$y' = \frac{\sinh(0.5y + x)}{1.5} + 0.5y \tag{7}$$

with the initial condition $y(0) = 0$ on the interval $[0; 0.2]$.

Solution. To choose the spacing compute the solution for the point $x = 0.1$ both with $h = 0.1$ and $h = 0.05$.

When computing with $h = 0.1$ we consecutively have

$$K_2^{(0)} = 0,$$

$$K_2^{(0)} = 0.1 \cdot \frac{\sinh 0.05}{1.5} = 0.003335,$$

$$K_3^{(0)} = 0.1\,\frac{\sinh(0.5 \cdot 0.001667 + 0.05)}{1.5} + 0.5 \cdot 0.001667 = 0.003475,$$

$$K_4^{(0)} = 0.1\,\frac{\sinh(0.5 \cdot 0.003475 + 0.1)}{1.5} + 0.5 \cdot 0.003475 = 0.006969.$$

Hence $\Delta y_0 = 0.003432$ and, consequently,

$$y_1 = y_0 + \Delta y_0 = 0.003432.$$

Then find the value $y(0.1)$ as the result of computations with $h = 0.05$ (see Table 72).

Table 72

Solving Equation (7) by the Runge-Kutta Method ($h = 0.05$)

x	y	$0.5y$	$0.5y+x$	$\sinh(0.5y+x)$	$f(x, y)$	$K=hf(x, y)$	Δy
0	0	0	0	0	0	0	
0.025	0	0	0.025	0.02500	0.01667	0.000834	0.000846
0.025	0.000417	0.000208	0.025208	0.02521	0.01702	0.000851	
0.05	0.000851	0.000426	0.050426	0.05045	0.03406	0.001703	
0.05	0.000846	0.000423	0.050423	0.05044	0.03405	0.001702	
0.075	0.001697	0.000848	0.075848	0.07592	0.05146	0.002573	0.002586
0.075	0.002132	0.001066	0.076066	0.07614	0.05183	0.002592	
0.1	0.003438	0.001719	0.101719	0.10190	0.06965	0.003482	
0.1	0.003432						

Since the obtained results coincide within the limits of the specified accuracy, go on computing both with $h = 0.1$ and $h = 0.2$. The results of further computations are entered in Tables 73 and 74.

The comparison of the results obtained for $h = 0.1$ and $h = 0.2$ shows that we may take $y(0.2) \approx 0.014158$ accurate to $5 \cdot 10^{-6}$, and

Table 73

Solving Equation (7) by the Runge-Kutta Method ($h=0.1$)

x	y	$0.5y$	$0.5y+x$	$sh(0.5y+x)$	$f(x, y)$	$K=hf(x, y)$	Δy
0.1	0.003432	0.001717	0.101717	0.10190	0.06965	0.006965	
0.15	0.006914	0.003457	0.153457	0.15406	0.10617	0.010617	0.010726
0.15	0.008740	0.004370	0.154370	0.15499	0.10770	0.010770	
0.2	0.014202	0.007101	0.207101	0.20858	0.14615	0.014615	
0.2	0.014158						

Table 74

Solving Equation (7) by the Runge-Kutta Method ($h=0.2$)

x	y	$0.5y$	$0.5y+x$	$sh(0.5y+x)$	$f(x, y)$	$K=hf(x, y)$	Δy
0.0	0	0	0	0	0	0	
0.1	0	0	0.1	0.10017	0.06678	0.013356	0.014155
0.1	0.006678	0.003339	0.103339	0.10352	0.07235	0.014470	
0.2	0.014470	0.007235	0.207235	0.20872	0.14638	0.029276	
0.2	0.014155						

that for further computations the spacing h should be doubled once again.

Note. The Runge-Kutta method is analogously used for solving systems of differential equations and equations of higher order, the latter being first reduced to systems of first-order differential equations.

Example 3. For the system of differential equations

$$\left.\begin{aligned} \frac{dx}{dt} &= -2x+5z, \\ \frac{dy}{dt} &= -(1-\sin t)\,x-y+3z, \\ \frac{dz}{dt} &= -x+2z \end{aligned}\right\} \qquad (8)$$

with the initial conditions $x(0)=2$, $y(0)=1$, $z(0)=1$ form a table of values of the functions $x(t)$, $y(t)$, $z(t)$ on the interval $[0; 0.3]$ for $h=0.1$, using the Runge-Kutta method.

Solution. All the calculations for the functions x, y, z are entered in Table 75 in the following succession.

Table 75

Solving System (8) by the Runge-Kutta Method

t	t	X	$K=hf$	qK	$\Delta X+\frac{1}{6}\sum qK$
0	0	2 1 1	0.1 0 0	0.1 0 0	0.08984 0.00497 —0.00500
	0.05	2.05 1 1	0.09 0.00525 —0.005	0.18 0.01050 —0.010	
	0.05	2.045 1.00262 0.99750	0.08975 0.00471 —0.00500	0.17950 0.00942 —0.01000	
	0.1	2.08975 1.00471 0.99500	0.07955 0.00992 —0.00998	0.07955 0.00992 —0.00998	
1	0.1	2.08984 1.00497 0.99500	0.07953 0.00988 —0.00998	0.07953 0.00988 —0.00988	0.06896 0.01456 —0.01494
	0.15	2.12960 1.00991 0.99001	0.06908 0.01480 —0.01496	0.13816 0.02960 —0.02992	
	0.15	2.12438 1.01237 0.98752	0.06888 0.01433 —0.01493	0.13777 0.02866 —0.02986	
	0.2	2.15872 1.01930 0.98007	0.05829 0.01911 —0.01986	0.05829 0.01911 —0.01986	
2	0.2	2.15880 1.01953 0.98006	0.05827 0.01907 —0.01987	0.05827 0.01907 —0.01987	0.04739 0.02313 —0.02473
	0.25	2.18794 1.02907 0.97012	0.04747 0.02346 —0.02477	0.09494 0.04692 —0.04954	
	0.25	2.18254 1.03126 0.96758	0.04733 0.02292 —0.02472	0.09466 0.04584 —0.04944	
	0.3	2.20613 1.04245 0.95534	0.03644 0.02694 0.02954	0.03644 0.02694 0.02954	

Denote: $f_1 = -2x + 5z$, $f_2 = -(1 - \sin t)x - y + 3z$, $f_3 = -x + 2z$.

(1) For $i = 0$ write down $t = 0$ and enter $x_0 = 2$, $y_0 = 1$, $z_0 = 1$ in the column X.

(2) Compute

$$f_1(t_0, x_0, y_0, z_0) = -4 + 5 = 1,$$
$$f_2(t_0, x_0, y_0, z_0) = -2 - 1 + 3 = 0,$$
$$f_3(t_0, x_0, y_0, z_0) = -2 + 2 = 0.$$

Multiplying the obtained values by h, we get the values K_1 for x, y, z, respectively. Write down them in the column K.

(3) Fill in the following three rows of the table for $t = 0.05$. In column X we write down $x_0 + \frac{K_{1x}^{(0)}}{2} = 2.05$, $y_0 + \frac{K_{1y}^{(0)}}{2} = 1$, $z_0 + \frac{K_{1z}^{(0)}}{2} = 1$.

(4) Compute f_1, f_2, f_3 for the written values of t, x, y, z and enter in the column K the values

$$K_{2x}^{(0)} = 0.09, \quad K_{2y}^{(0)} = 0.00525, \quad K_{2z}^{(0)} = -0.005.$$

(5) Fill in next three rows of the table for $t = 0.05$. In column X we write down

$$x_0 + \frac{K_{2x}^{(0)}}{2} = 2.045, \quad y_0 + \frac{K_{2y}^{(0)}}{2} = 1.00262, \quad z_0 + \frac{K_{2z}^{(0)}}{2} = 0.99750$$

and perform computations in an analogous way.

Enter the values $K_1^{(0)}$, $2K_2^{(0)}$, $2K_3^{(0)}$, $K_4^{(0)}$ for x, y, z in the column qK.

Then, using these numbers, we find

$$\Delta x_0 = \frac{1}{6}(K_{1x}^{(0)} + 2K_{2x}^{(0)} + 2K_{3x}^{(0)} + K_{4x}^{(0)}) = 0.08984$$

and, analogously, $\Delta y_0 = 0.00497$, $\Delta z_0 = -0.00500$. Then

$$x_1 = x_0 + \Delta x_0 = 2.08984, \quad y_1 = y_0 + \Delta y_0 = 1.00497,$$
$$z_1 = z_0 + \Delta z_0 = 0.99500.$$

Enter the obtained values x_1, y_1, z_1 in Table 75 for $i = 1$, and go on performing computations according to the scheme considered.

Problems

1. Using the Runge-Kutta method with $h = 0.2$, find the solutions of the given equations and systems on the indicated interval $[a, b]$.

(a) $y' = y - x$, $y(0) = 1.5$, $a = 0$, $b = 1$,

(b) $y' = \frac{y}{x} - y^2$, $y(1) = 1$, $a = 1$, $b = 2$,

c) $\left.\begin{aligned} y' &= z+1, \\ z' &= y-x, \end{aligned}\right\}$ $y(0)=1, \quad z(0)=1, \quad a=0, \quad b=1.$

Applying the Runge-Kutta method with $h=0.1$, find on the interval $[0;\ 0.3]$ the solutions of the following differential equations and systems of differential equations for the given initial conditions.

2. $x'=\frac{\cos bt}{a+x^2}$, $x(0)=0$, $a=1.0+0.4\cdot n$, $n=0, 1, 2, 3, 4, 5$, $b=1.0+0.8\cdot k$, $k=0, 1, 2$.

3. $x'=\frac{a}{t^2+x^2+b}$, $x(0)=0$, $a=1.0+0.4\cdot n$, $n=0, 1, \ldots, 5$, $b=1.0+0.4\cdot k$, $k=0, 1, \ldots, 5$.

4. $x'=e^{-at}(x^2+b)$, $x(0)=0$, $a=1.0+0.4\cdot n$, $n=0, 1, \ldots, 5$, $b=1.0+0.4\cdot k$, $k=0, 1, \ldots, 5$.

5. $x'=\cos(at+x)+t-x$, $x(0)=0$, $a=1.0+0.4\cdot k$, $k=0, 1, \ldots, 5$.

6. $x'=1-\sin(at+x)+\frac{bx}{2+t}$, $x(0)=0$, $a=1.0+0.4\cdot k$, $k=0, 1, \ldots, 5$, $b=1.0+0.8\cdot n$, $n=0, 1, 2$.

7. $x'=\frac{\cos x}{a+t}+x^2$, $x(0)=0$, $a=1.0+0.4\cdot k$, $k=0, 1, \ldots, 5$.

8. $x'=1+ax\sin t-x^2$, $x(0)=0$, $a=1.0+0.4\cdot k$, $k=0, 1, \ldots, 5$.

9. $\left.\begin{aligned} x' &= \cos(x+ay)+b, \\ y' &= \frac{a}{t+bx^2}+t+1, \end{aligned}\right\}$ $x(0)=1$, $y(0)=0.05$, $a=2.0+0.5\cdot n$, $n=0, 1, 2, 3$, $b=2.0+0.5\cdot k$, $k=0, 1, \ldots, 5$.

10. $\left.\begin{aligned} x' &= \sin(ax^2)+t+y, \\ y' &= t+x-by^2+1, \end{aligned}\right\}$ $x(0)=1$, $y(0)=0.5$, $a=2.0+0.5\cdot n$, $n=0, 1, 2, 3$, $b=2.0+0.5\cdot k$, $k=0, 1, \ldots, 5$.

11. $\left.\begin{aligned} x' &= \sqrt{t^2+ax^2}+y, \\ y' &= \cos(by)+x, \end{aligned}\right\}$ $x(0)=0.5$, $y(0)=1$, $a=2.0+0.5\cdot n$, $n=0, 1, 2, 3$, $b=2.0+0.5\cdot k$, $k=0, 1, \ldots, 5$.

12. $\left.\begin{aligned} x' &= e^{-(x^2+y^2)}+at, \\ y' &= bx^2+y, \end{aligned}\right\}$ $x(0)=0.5$, $y(0)=1$, $a=2.0+0.5\cdot n$, $n=0, 1, 2, 3$, $b=2.0+0.5\cdot k$, $k=0, 1, \ldots, 5$.

13. $\left.\begin{aligned} x' &= \ln(bt+\sqrt{a^2t^2+y^2}), \\ y' &= \sqrt{a^2t^2+x^2}, \end{aligned}\right\}$ $x(0)=1$, $y(0)=0.5$, $a=2.0+0.5\cdot n$, $n=0, 1, 2, 3$, $b=2.0+0.5\cdot k$, $k=0, 1, \ldots, 5$.

14. Solve Problem 11 of Sec. 8.4 by the Runge-Kutta method with $h_1=100$ and $h_2=50$.

15. Solve Problem 12 of Sec. 8.4 by the Runge-Kutta method with $h=10$.

16. Solve Problem 13 of Sec. 8.4 by the Runge-Kutta method with $h=10$.

8.8 ADAMS' METHOD

Let for the equation

$$y'=f(x,\ y) \tag{1}$$

with the initial condition $y(x_0)=y_0$ three consecutive values of the required function (the "initial interval") be found by one of the above considered methods

$$y_1=y(x_1)=y(x_0+h),\quad y_2=y(x_2)=y(x_0+2h),$$
$$y_3=y(x_3)=y(x_0+3h).$$

With the aid of these values we compute the quantities

$$q_0=hy'_0=hf(x_0,\ y_0),\quad q_1=hy'_1=hf(x_1,\ y_1),$$
$$q_2=hy'_2=hf(x_2,\ y_2),\quad q_3=hy'_3=hf(x_3,\ y_3).$$

Write down the numbers x_k, y_k, y'_k, q_k $(k=0,\ 1,\ 2,\ 3)$ in Table 76 and compute the finite differences of the quantity q (the number above the stepped line in Table 76).

Table 76

Adams' Method Scheme

k	x_k	y_k	$\Delta y_k=y_{k+1}-y_k$	$y'_k=f(x_k,\ y_k)$	$q_k=hy'_k$	$\Delta q_k=q_{k+1}-q_k$	$\Delta^2 q_k$	$\Delta^3 q_k$
0	x_0	y_0	Δy_0	$f(x_0,\ y_0)$	q_0	Δq_0	$\Delta^2 q_0$	$\Delta^3 q_0$
1	x_1	y_1	Δy_1	$f(x_1,\ y_1)$	q_1	Δq_1	$\Delta^2 q_1$	$\Delta^3 q_1$
2	x_2	y_2	Δy_2	$f(x_2,\ y_2)$	q_2	Δq_2	$\Delta^2 q_2$	$\Delta^3 q_2$
3	x_3	y_3	Δy_3	$f(x_3,\ y_3)$	q_3	Δq_3	$\Delta^2 q_3$	
4	x_4	y_4	Δy_4	$f(x_4,\ y_4)$	q_4	Δq_4		
5	x_5	y_5	Δy_5	$f(x_5,\ y_5)$	q_5			
6	x_6	y_6						

The Adams method (see [2], [13], [20]) consists in expanding the difference table with the aid of the formula

$$\Delta y_k=q_k+\frac{1}{2}\Delta q_{k-1}+\frac{5}{12}\Delta^2 q_{k-2}+\frac{3}{8}\Delta^3 q_{k-3}\quad (k=3,\ 4,\ \ldots), \tag{2}$$

which is called *Adams' extrapolation formula* and is used for "predicting" the value of $y_{k+1}=y_k+\Delta y_k$. Let us denote the "predicted" value computed by this formula by y_{k+1}^{pred}. The value of Δy_k obtained by formula (2) has to be specified. To this end we have to enter the values x_{k+1}, y_{k+1}, y'_{k+1}, q_{k+1} in the table, supplement the difference table, and then check the computation by the "correction" formula

$$\Delta y_k=q_k+\frac{1}{2}\Delta q_k-\frac{1}{12}\Delta^2 q_{k-1}-\frac{1}{24}\Delta^3 q_{k-2}, \tag{3}$$

which is called *Adams' interpolation formula*. We denote the value specified by formula (3) by y_{k+1}^{corr}.

The formulas (2) and (3) are of a very high accuracy, yielding an error of order $O(h^4)$, but the formulas for estimating the error are rather complicated.

For practical purposes the following is usually taken into consideration. The error of the more accurate correction formula (3) constitutes about $\frac{1}{14}$ the difference between the values of Δy_k calculated by formulas (2) and (3). Therefore, if this difference does not considerably exceed the permissible computational error, then the spacing h is considered to be chosen adequately, and the computation is continued with the chosen spacing. But if at a certain stage of computation the mentioned difference increases materially (and the calculations themselves are void of errors!), then the spacing h should be decreased (it is usually reduced by half).

Filling in Table 76.

(1) Write down the numbers x_k, y_k, y'_k, q_k $(k=0, 1, 2, 3)$, and compute the differences Δq_k $(k=0, 1, 2)$, $\Delta^2 q_k$ $(k=0, 1)$, $\Delta^3 q_0$.

(2) Using the numbers q_3, Δq_2, $\Delta^2 q_1$, $\Delta^3 q_0$ placed diagonally in the table, determine by formula (2) for $k=3$

$$\Delta y_3=q_3+\frac{1}{2}\Delta q_2+\frac{5}{12}\Delta^2 q_1+\frac{3}{8}\Delta^3 q_0.$$

(3) Compute $x_4=x_3+h$, $y_4=y_3+\Delta y_3$.

(4) Enter the values x_4, y_4, find $y'_4=f(x_4, y_4)$, $q_4=hy'_4$, and supplement the difference table with the values of Δq_3, $\Delta^2 q_2$, $\Delta^3 q_1$.

(5) Using the obtained values of the differences q, specify the quantity Δy_3 by formula (3) for $k=3$:

$$\Delta y_3=q+\frac{1}{2}\Delta q_3-\frac{1}{12}\Delta^2 q_2-\frac{1}{24}\Delta^3 q.$$

(6) If the corrected value of Δy_3 differs from its predicted value by several units of the lower retained order, then introduce the corresponding corrections in the values of Δy_3 and y_4, check to see that the corrections do not affect considerably the value of q_4,

and continue the computation with the chosen spacing. If otherwise, choose a smaller spacing.

The computations for $k=4, 5, \ldots$ are accomplished in a similar way.

When carrying out computational work on a computer, it is more convenient to use transformed Adams' formulas, which express y_{k+1} not in terms of the difference Δq, but directly in terms of the quantity q. Thus, we obtain the *extrapolation formula of Adams* in the form

$$y_{k+1}=y_k+\frac{h}{24}(55y'_k-59y'_{k-1}+37y'_{k-2}-9y'_{k-3}) \tag{4}$$

and the *interpolation formula of Adams* in the form

$$y_{k+1}=y_k+\frac{h}{24}(9y'_{k+1}+19y'_k-5y'_{k-1}+y'_{k-2}). \tag{5}$$

The Adams method is readily applicable to systems of differential equations, as also to differential equations of the nth order.

Suppose we have a system of two equations

$$\left.\begin{aligned} y'&=f_1(x, y, z),\\ z'&=f_2(x, y, z). \end{aligned}\right\} \tag{6}$$

Adams' extrapolation formulas for this system are written in the following way:

$$\left.\begin{aligned} \Delta y_k&=p_k+\frac{1}{2}\Delta p_{k-1}+\frac{5}{12}\Delta^2 p_{k-2}+\frac{3}{8}\Delta^3 p_{k-3},\\ \Delta z_k&=q_k+\frac{1}{2}\Delta q_{k-1}+\frac{5}{12}\Delta^2 q_{k-2}+\frac{3}{8}\Delta^3 q_{k-3}, \end{aligned}\right\} \tag{7}$$

where

$$p_k=hy'_k=hf_1(x_k, y_k, z_k), \quad q_k=hz'_k=hf_2(x_k, y_k, z_k).$$

Adams' interpolation formulas for the considered system are written analogously.

Example 1. Using Adams' method, find on the interval $[0; 0.5]$ the solution of the differential equation

$$y'=\frac{\sinh(0.5y+x)}{1.5}+0.5y \tag{8}$$

with the initial condition $y(0)=0$; take $h=0.05$.

Solution. In Example 2 of Sec. 8.7 the values of the required function were computed by the Runge-Kutta method for $x_1=0.05$, $x_2=0.10$. Analogously, we can calculate the value of the sought-for function for $x_3=0.15$ as well. Let us take advantage of these results and continue the computations by formula (2); entering the yielded results in two tables, Table 77 being the basic difference table, and Table 78 an auxiliary table for computing the right-hand member of equation (8).

Table 77

Solving Equation (8) by Adams' Method

k	x_k	y_k	Δy_k	$q_k = hf(x_k, y_k)$	Δq_k	$\Delta^2 q_k$	$\Delta^3 q_k$
0	0	0		0	1702	78	7
1	0.05	0.000846		0.001702	1780	85	9
2	0.10	0.003432		0.003482	1865	94	11
3	0.15	0.007838	06318	0.005347	1959	105	11
4	0.20	0.014156	08329	0.007306	2064	116	13
5	0.25	0.022485	10451	0.009370	2180	129	13
6	0.30	0.032936	12692	0.011550	2309	142	17
7	0.35	0.045628	15070	0.013859	2451	159	
8	0.40	0.060698	17603	0.016310	2610		
9	0.45	0.078301	20295	0.018920			
10	0.50	0.098596					

Table 78

Computing the Right-Hand Member of Equation (8)

k	x_k	y_k	$0.5y_k$	$0.5y_k + x_k$	$\sinh(0.5y_k + x_k)$	$f(x_k, y_k)$
4	0.20	0.014156	0.007078	0.207078	0.20856	0.14612
5	0.25	0.022485	0.011242	0.261242	0.26422	0.18739
6	0.30	0.032936	0.016468	0.316468	0.32178	0.23099
7	0.35	0.045628	0.022814	0.378814	0.38151	0.27715
8	0.40	0.060698	0.030349	0.430349	0.44376	0.32619
9	0.45	0.078301	0.039150	0.489150	0.50889	0.37841

Filling in the tables:

(1) Enter in Table 77 the values $x_0 = 0$, $x_1 = 0.05$, $x_2 = 0.10$, $x_3 = 0.15$ and the corresponding values of y_k $(k = 0, 1, 2, 3)$, find $f(x_k, y_k)$, q_k, and form a difference table.

(2) By formula (2) for $k = 3$ we find

$$\Delta y_3 = 0.005347 + \frac{1}{2} \cdot 0.001865 + \frac{5}{12} \cdot 0.000085 + \frac{3}{8} \cdot 0.000007 = 0.006318.$$

(3) Compute $y_4 = 0.007838 + 0.006318 = 0.014156$.

(4) Entering the values x_4, y_4 in Table 78, we find

$$y'_4 = f(x_4, y_4) = \frac{2}{3} \sinh(0.5 \cdot 0.014156 + 0.2) + 0.5 \cdot 0.014156 = 0.14612;$$

then

$$q_4 = hy'_4 = 0.007306.$$

Write down the result obtained and compute the differences Δq_3, $\Delta^2 q_2$, $\Delta^3 q_1$.

(5) Compute the corrected value by formula (3)

$$\Delta y_3 = 0.005347 + \frac{1}{2}\,0.001959 - \frac{1}{12}\,0.000094 - \frac{1}{24}\,0.000009 = 0.006318.$$

Since the corrected value of Δy_3 coincides with its predicted value, we continue to carry out the computation with the chosen spacing and without resorting to a further correction.

(6) By formula (2) for $k=4$ we find

$$\Delta y_4 = 0.007306 + \frac{1}{2}\,0.001959 + \frac{5}{12}\,0.000094 + \frac{3}{8}\,0.000009 = 0.008329,$$

and so on.

Note. Here, as in solving Example 2 of Sec. 8.7, we might have conducted the computations with a greater spacing, which is, in particular, confirmed by the small values of the differences $\Delta^3 q$.

Example 2. Using Adams' method, find on the interval [0; 0.5] the solution of the equation

$$y' = 0.25y^2 + x^2 \tag{9}$$

with the initial condition $y(0) = -1$ accurate to 10^{-5}.

Solution. In Example 7.1 the values of the required function were computed by the Runge-Kutta method with $h=0.1$ for $x_1 = 0.1$, $x_2 = 0.2$, $x_3 = 0.3$. The further computations will be carried out on a computer by formula (4) and corrected by formula (5), the yielded results being entered in two tables. Table 79 contains the computations by formula (4) corrected by formula (5), Table 80, the results of computation of the right-hand member.

Table 79

Solving Equation (9) by Adams' Method

k	x_k	y_k	y'_k	$\frac{\alpha_k}{24}$	$\frac{\beta_k}{24}$	$h\frac{\alpha_k}{24}$	$h\frac{\beta_k}{24}$
0	0.0	−1	0.25				
1	0.1	−0.97528	0.24779				
2	0.2	−0.94978	0.26552				
3	0.3	−0.92154	0.30232	0.32834	0.32840	0.03283	0.03284
4	0.4	−0.88871	0.35745	0.39237	0.39246	0.03924	0.03925
		−0.88870					
5	0.5	−0.84946	0.43040				
		−0.84946					

Table 80

Computing the Right-Hand Member of Equation (9)

k	x	x^2	y	$0.25y^2$	$f(x, y)=0.25y^2+x^2$
0	0	0	—1	0.25	0.25
1	0.1	0.01	—0.97528	0.23779	0.24779
2	0.2	0.04	—0.94978	0.22552	0.26552
3	0.3	0.09	—0.92154	0.21232	0.30232
4	0.4	0.16	—0.88871 —0.88870	0.19745	0.35745
5	0.5	0.25	—0.84946 —0.84946	0.18040	0.43040

In Table 79 α_k and β_k denote the sums

$$\alpha_k = 55y'_k - 59y'_{k-1} + 37y'_{k-2} - 9y'_{k-3}, \quad \beta_k = 9y'_{k+1} + 19y'_k - 5y'_{k-1} + y'_{k-2}.$$

Filling in the tables:

(1) Enter in Table 80 the values $x_0 = 0$, $x_1 = 0.1$, $x_2 = 0.2$, $x_3 = 0.3$ and the corresponding values of y_k ($k = 0, 1, 2, 3$). Using them, compute $y'_k = f(x_k, y_k)$.

(2) Compute the quantity $\frac{\alpha_3}{24} = \frac{1}{24}(55y'_3 - 59y'_2 + 37y'_1 - 9y'_0) = 0.32834$, enter it in Table 79 and by formula (4) for $k = 4$ we find

$$y_4^{\text{pred.}} \quad y_3 + h\frac{\alpha_3}{24} = -0.92154 + 0.1 \cdot 0.32834 = -0.88871.$$

(3) Enter the values x_4, y_4 in Table 80, and, using them, compute $y'_4 = f(x_4, y_4) = 0.35745$.

(4) Compute the quantity

$$\frac{\beta_3}{24} = \frac{1}{24}(9y'_4 + 19y'_3 - 5y'_2 + y'_1) = 0.32840,$$

enter it in Table 79 for $k = 3$ and specify the value of y_4 by formula (8.5):

$$y_4^{\text{corr}} \quad y_3 + h\frac{\beta_3}{24} = -0.92154 + 0.1 \cdot 0.32840 = -0.88870.$$

(5) Since the obtained values of $y_4^{\text{pred.}}$ and y_4^{corr} differ only by 10^{-5}, we introduce a correction in the value of y_4 in Table 79 and 80

$$y_4 = -0.88870.$$

(6) Write down the obtained value $y_4 = y_4^{\text{corr.}}$ in Table 80, and make sure that the correction does not change the value of $y'_4 = f(x_4, y_4)$.

The computations of y_5 are performed analogously.

Example 3. Given a system of equations

$$\left.\begin{aligned} y' &= \cos(y + 1.1z) + 1, \\ z' &= \frac{1}{x + 2.1y^2} + x + 1 \end{aligned}\right\} \tag{10}$$

with the initial conditions $y(0) = \pi$, $z(0) = 0$. Using Adams' method, find the solution of this system on the interval $[0; 0.6]$, taking a spacing $h = 0.1$ and considering as known the values of the functions $y(x)$ and $z(x)$ for $x_1 = 0.1$, $x_2 = 0.2$, $x_3 = 0.3$; they are equal to $y(0.1) = 3.14184$; $y(0.2) = 3.14364$; $y(0.3) = 3.14903$; $z(0.1) = 0.10981$; $z(0.2) = 0.22960$, $z(0.3) = 0.35934$, respectively.

Solution. We shall look for the values of the functions $y(x)$, $z(x)$ for $x_4 = 0.4$, $x_5 = 0.5$, $x_6 = 0.6$ with the aid of formulas (8.7), denoting

$$f_1(x, y, z) = \cos(y + 1.1z) + 1, \quad f_2(x, y, z) = \frac{1}{x + 2.1y^2} + x + 1.$$

All the computations are contained in two tables: Table 81 is the basic table of the Adams method, Table 82 being an auxiliary table for calculating the right-hand members.

Table 81

Solving System (10) by Adams' Method

k	x_k	y_k	Δy_k	p_k	Δp_k	$\Delta^2 p_k$	$\Delta^3 p_k$
0	0	3.14159		0	73	176	53
1	0.1	3.14184		0.00073	249	229	63
2	0.2	3.14364		0.00322	478	292	66
3	0.3	3.14903	0.01154	0.00800	770	358	
4	0.4	3.16057	0.02101	0.01570	1128		
5	0.5	3.18158	0.03446	0.02708			
6	0.6	3.21604					

k	z_k	Δz_k	q_k	Δq_k	$\Delta^2 q_k$	$\Delta^3 q_k$
0	0		0.10482	998	—1	0
1	0.10981		0.11480	997	—1	0
2	0.22960		0.12477	996	—1	2
3	0.35934	0.13971	0.13473	995	—3	
4	0.49905	0.14965	0.14468	992		
5	0.64870	0.15955	0.15460			
6	0.80825					

Table 82

Computing the Right-Hand Members of System (10)

k	x	y^2	$\frac{1}{x+2.1y^2}$	$z'=f_2(x, y, z)$	$y+1.1z$	$y'=f(x, y, z)$
0	0.0	9.8696	0.04824	1.04824	3.14159	0
1	0.1	9.8712	0.04801	1.14801	3.26264	0.00732
2	0.2	9.8825	0.04773	1.24773	3.39620	0.03224
3	0.3	9.9164	0.04734	1.34734	3.54433	0.08001
4	0.4	9.9896	0.04478	1.44678	3.70954	0.15700
5	0.5	10.1233	0.04596	1.54596	3.89524	0.27080

Filling in the tables.

(1) Enter the values of $x_k=kh$ and the corresponding values of y_k and z_k $(k=0, 1, 2, 3)$.

(2) Compute $y'_k=f_1(x_k, y_k, z_k)$, $z'_k=f_2(x_k, y_k, z_k)$ $(k=0, 1, 2, 3)$.

(3) Find the numbers $p_k=hy'_k$, $q=hz'_k$ $(k=0, 1, 2, 3)$, enter them in Table 81 and form the differences Δp, $\Delta^2 p$, $\Delta^3 p$, Δq, $\Delta^2 q$, $\Delta^3 q$.

(4) Using the obtained differences, by formulas (7) for $k=3$ we get

$$\Delta y_3=0.00800+\frac{1}{2}\cdot 0.00478+\frac{5}{12}\cdot 0.00229+\frac{3}{8}\cdot 0.00053=0.01154,$$

$$\Delta z_3=0.13473+\frac{1}{2}\cdot 0.00996+\frac{5}{12}(-0.00001)=0.13971.$$

(5) Determine

$$y_4=y_3+\Delta y_3=3.16057, \quad z_4=z_3+\Delta z_3=0.49905.$$

Write down the obtained values in Table 8.6 and 8.7 and, using them, compute $y'_5=f_1(x_5, y_5, z_5)$, $z'_5=f_2(x_5, y_5, z_5)$.

(6) Find the numbers $p_5=hy'_5$, $q_5=hz'_5$, enter them in Table 81, supplement the difference table, and proceed with the computations by formulas (7) for $k=4$.

Here we do not specify the obtained values by the correction formulas, since the corrected values of the differences coincide with the predicted ones within the limits of computational accuracy. For instance, for $k=4$

$$\Delta y_4^{\text{corr.}}=0.01570+\frac{1}{2}\cdot 0.01128-\frac{1}{12}\cdot 0.00358-\frac{1}{24}\cdot 0.00066=0.02101,$$

$$\Delta z_4^{\text{corr.}}=0.14468+\frac{1}{2}\cdot 0.00992-\frac{1}{12}(-0.00003)=0.14964.$$

Problems

1. Using the Adams method, find to within 10^{-2} the numerical solutions of the following differential equations and systems of

equations for the given values of the argument. Find the "initial interval" by the Runge-Kutta method.

(a) $y' = x + y$, $y(0) = 1$, find $y(0.5)$,

(b) $y' = x^2 + y$, $y(0) = 1$, find $y(1)$,

(c) $y' = 2y - 3$, $y(0) = 1$, find $y(0.5)$,

(d) $\left.\begin{aligned} y' &= -x + 2y + z, \\ z' &= x + 2y + 3z, \end{aligned}\right\}$ $y(0) = 2$, $z(0) = -2$, find y and z for $x = 0.5$,

(e) $\left.\begin{aligned} y' &= -3y - z \\ z' &= y - z, \end{aligned}\right\}$ $y(0) = 2$, $z(0) = -1$, find y and z for $x = 0.5$.

2. Using Adams' method, solve numerically the following differential equations on the interval $[a, b]$ accurate to 10^{-4}. Find the "initial interval" by the Runge-Kutta method.

(a) $y' = xy^3 - y$, $y(0) = 1$, $a = 0$, $b = 1$, (b) $y' = y^2 e^x = 2y$, $y(0) = 1$, $a = 0$, $b = 1$, (c) $y' = \dfrac{1}{y^2 - x}$, $y(1) = 0$, $a = 1$, $b = 2$.

Using the Adams method, expand by two spacings the table of numerical solutions of the following equations with the initial conditions $x(0) = 0$. Take $h = 0.1$. Find the "initial interval" by the Runge-Kutta method (see Sec. 8.7, Problems 2 to 8).

3. $x' = \dfrac{\cos bt}{a + x^2}$, $a = 1.0 + 0.4 \cdot n$, $n = 0, 1, \ldots, 5$, $b = 1.0 + 0.8 \cdot k$, $k = 0, 1, 2$.

4. $x' = \dfrac{a}{t^2 + x^2 + b}$, $a = 1.0 + 0.4 \cdot n$, $n = 0, 1, \ldots, 5$, $b = 1.0 + 0.4 \cdot k$, $k = 0, 1, \ldots, 5$.

5. $x' = e^{-at}(x^2 + b)$, $a = 1.0 + 0.4 \cdot n$, $n = 0, 1, \ldots, 5$; $b = 1.0 + 0.4 \cdot k$, $k = 0, 1, \ldots, 5$.

6. $x' = \cos(at + x) + (t - x)$, $a = 1.0 + 0.4 \cdot k$, $k = 0, 1, 2, \ldots, 5$.

7. $x' = 1 - \sin(at + x) + \dfrac{bx}{2 + t}$, $a = 1.0 + 0.4 \cdot k$, $k = 0, 1, 2, \ldots, 5$, $b = 1.0 + 0.8n$, $n = 0, 1, 2$.

8. $x' = \dfrac{\cos x}{a + t} + x^2$, $a = 1.0 + 0.4 \cdot k$, $k = 0, 1, 2, \ldots, 5$.

9. $x' = 1 + ax \sin t - x^2$, $a = 1.0 + 0.4 \cdot k$, $k = 0, 1, 2, \ldots, 5$.

Using the Adams method, expand by two spacings the table of numerical solutions of the following systems with the initial conditions specified. Take $h = 0.1$. Find the "initial interval" by the Runge-Kutta method (see Sec. 8.7, Problems 9 to 13).

10. $\left.\begin{aligned} x' &= \cos(x + ay) + b, \\ y' &= \dfrac{a}{t + bx^2} + t + 1, \end{aligned}\right\}$ $\begin{aligned} &x(0) = 1, \ y(0) = 0.05, \\ &a = 2.0 + 0.5 \cdot n, \ n = 0, 1, 2, 3, \\ &b = 2.0 + 0.5 \cdot k, \ k = 0, 1, 2, \ldots, 5. \end{aligned}$

11. $\left.\begin{aligned} x' &= \sin(ax^2) + t + y, \\ y' &= t + x - by^2 + 1, \end{aligned}\right\}$ $\left.\begin{aligned} &x(0) = 1,\ y(0) = 0.5,\ a = 2.0 + 0.5 \cdot n, \\ &n = 0, 1, 2, 3,\ b = 2.0 + 0.5 \cdot k, \\ &k = 0, 1, 2, \ldots, 5. \end{aligned}\right.$

12. $\left.\begin{aligned} x' &= \sqrt{t^2 + ax^2} + y, \\ y' &= \cos(by) + x, \end{aligned}\right\}$ $\left.\begin{aligned} &x(0) = 0.5,\ y(0) = 1,\ a = 2.0 + 0.5 \cdot n, \\ &n = 0, 1, 2, 3,\ b = 2.0 + 0.5 \cdot k, \\ &k = 0, 1, 2, \ldots, 5. \end{aligned}\right.$

13. $\left.\begin{aligned} x' &= e^{-(x^2+y^2)} + at, \\ y &= bx^2 + y, \end{aligned}\right\}$ $\left.\begin{aligned} &x(0) = 0.5,\ y(0) = 1,\ a = 2.0 + 0.5 \cdot n, \\ &n = 0, 1, 2, 3,\ b = 2.0 + 0.5 \cdot k, \\ &k = 0, 1, 2, \ldots, 5. \end{aligned}\right.$

14. $\left.\begin{aligned} x' &= \ln(bt + \sqrt{a^2t^2 + y^2}) \\ y' &= \sqrt{a^2t^2 + x^2}, \end{aligned}\right\}$ $\left.\begin{aligned} &x(0) = 1,\ y(0) = 0.5, \\ &a = 2.0 + 0.5 \cdot n,\ n = 0, 1, 2, 3, \\ &b = 2.0 + 0.5 \cdot k,\ k = 0, 1, 2, \ldots, 5. \end{aligned}\right.$

15. Using the Adams method, find the numerical solutions of the following differential equations for $x = 0.2$. Use the values of the "initial interval" obtained in Problem 14 of Sec. 8.2.

(a) $y' = y^2 + x^2,\ y(0) = 0.5,$

(b) $\left.\begin{aligned} y' &= xy + z, \\ z' &= y - z, \end{aligned}\right\}$ $y(0) = 0,\ z(0) = 1,$

(c) $\left.\begin{aligned} y' &= x + z^2, \\ z' &= xy, \end{aligned}\right\}$ $y(0) = 1,\ z(0) = 1.$

8.9 MILNE'S METHOD

Let for equation (1) of Sec. 8.8, beside the initial condition $y(x_0) = y_0$ the "initial interval" be known, i.e. the values of the required function $y(x_i) = y_i$ for the points $x_i = x_0 + ih$, $i = 1, 2, 3$ (they can be found by one of the above considered methods). The subsequent values of y_i for $i = 4, 5, \ldots$ are determined in the following way (see [13]). For prediction *Milne's first formula* is used

$$y_i^{\text{pred}} = y_{i-4} + \frac{4h}{3}(2y'_{i-3} - y'_{i-2} + 2y'_{i-1}). \tag{1}$$

Using $y_i^{\text{pred.}}$, find $y'_i = f(x_i, y_i^{\text{pred.}})$ and carry out correction by *Milne's second formula*

$$y_i^{\text{corr.}} = y_{i-2} + \frac{h}{3}(y'_{i-2} + 4y'_{i-1} + y'_i). \tag{2}$$

The absolute error ε_i of the more correct value $y_i^{\text{corr.}}$ is approximately determined by the formula

$$\varepsilon_i \approx \frac{1}{29} |y_i^{\text{corr.}} - y_i^{\text{pred.}}|. \tag{3}$$

This formula enables us to check the accuracy of the obtained results step by step. If we have to find the required solution accurate to ε, and it turns out that $\varepsilon_i \leqslant \varepsilon$, then we may put

$$y_i \approx y_i^{\text{corr.}}$$

and pass to computing y_{i+1}. Otherwise the spacing h should be reduced.

The remainder terms of formulas (1), (2) have the order $O(h^5)$ for each spacing and $O(h^4)$ over the entire interval $[x_0, x_n]$.

Milne's method is used to find approximate solutions of systems of differential equations of the first order, as well as equations of higher orders which first should be transformed to such systems.

Example 1. Using Milne's method, find to within $3 \cdot 10^{-4}$ on the interval [0, 1] the solution of the equation

$$xy'' + y' + xy = 0 \tag{4}$$

with the initial condition $y(0) = 1$, $y'(0) = 0$.

Solution. Transform the equation (4) into a system. In this case it is favourable to use the substitution $xy' = z$. Finally, we get the system

$$\left.\begin{aligned} y' &= \frac{z}{x}, \\ z' &= -xy \end{aligned}\right\} \tag{5}$$

with the initial conditions $y(0) = 1$, $z(0) = 0$.

Take $h = 0.2$. To get the initial interval make use of the answer to Problem 12 of Sec. 8.2, where the solution of equation (4) is given in the form of a series. Take $y(x)$ in the form of the interval of this series retaining only the terms which for the point $x_3 = 0.6$ yield a value exceeding 10^{-4}:

$$y(x) \approx 1 - \frac{x^2}{4} + \frac{x^4}{64}.$$

Then $z'(x) \approx -x + \frac{x^3}{4} - \frac{x^5}{64}$; retaining only the terms exceeding 10^{-4} at the point $x_3 = 0.6$, we get

$$z(x) \approx -\frac{x^2}{2} + \frac{x^4}{16} - \frac{x^6}{384}.$$

Filling in the table:

(1) Form a table of four-digit values of $y(x)$ and $z(x)$ for $x_1 = 0.2$, $x_2 = 0.4$, $x_3 = 0.6$ (Table 83).

(2) Compute $y_i' = z_i/x_i$, $z_i' = -x_i y_i$ $(i = 0, 1, 2, 3)$.

(3) Compute (on a computer) the predicted "differences y" and

Table 83

Solving System (5) by Milne's Method

i	x_i	y_i	z_i	y'_i	z'_i	differences y	differences z	
0	0	1	0	0	0			
1	0.2	0.9900	—0.01990	—0.0995	—0.1980			
2	0.4	0.9604	—0.07841	—0.1960	—0.3842			
3	0.6	0.9120	—0.17202	—0.2867	—0.5472			
4	0.8	0.8463	—0.2950	—0.3688	—0.6770	—0.1537	—0.2950	prediction
		0.8463	—0.2951	—0.3689	—0.6770	—0.1141	—0.2167	correction
5	1.0	0.7652	—0.4400	—0.4400	—0.7652			prediction
		0.7652	—0.4400					correction

"differences z" by formula (1)

$$y_4^{\text{pred.}} - y_0 = \frac{4h}{3}(2y'_1 - y'_2 + 2y'_3) = -0.1537,$$

$$z_4^{\text{pred.}} - z_0 = \frac{4h}{3}(2z'_1 - z'_2 + 2z'_3) = -0.2950.$$

Enter the results in row $i = 4$.

(4) Compute the predicted values of the required functions at the point $x_4 = 0.8$:

$$y_4^{\text{pred.}} = 1 - 0.1537 = 0.8463,$$
$$z_4^{\text{pred.}} = 0 - 0.2950 = -0.2950.$$

Write them down in the row $i = 4$ and find $y'_4 = \frac{z_4^{\text{pred.}}}{x^4}$, $z'_4 = -x_4 y_4^{\text{pred.}}$.

(5) Compute (on a computer) the corrected "differences y" and "differences z" by formula (2):

$$y_4^{\text{corr.}} - y_2 = \frac{h}{3}(y'_2 - 4y'_3 + y'_4) = -0.1141,$$

$$z_4^{\text{corr.}} - z_2 = \frac{h}{3}(z'_2 - 4z'_3 + z'_4) = -0.2167.$$

Enter the results in the row $i = 4$.

(6) Compute the corrected values of the required functions at the point $x_4 = 0.8$:

$$y_4^{\text{corr.}} = 0.9604 - 0.1141 = 0.8463,$$
$$z_4^{\text{corr.}} = -0.0784 - 0.2167 = -0.2951.$$

(7) Since the difference between the predicted and corrected values

does not exceed 10^{-4}, we put

$$y_4 = 0.8463, \quad z_4 = -0.2951$$

and proceed with the calculations for $i = 5$.

Note. Milne's method is void of stability, therefore it is recommended to be used only when the supposed number of pitches is small.

Problems

Using Milne's method, find (to within 10^{-4}) the solutions of the following equations on the interval $[a, b]$ for the given initial conditions. Find the "initial interval" by one of the above considered methods.

1. $y' = -\frac{y}{x} - \frac{2y^2}{\alpha} \ln x$, $y(1) = 1$, $a = 1$, $b = 2$, $\alpha = 0.5 + 0.25 \cdot k$, $k = 0, 1, 2, \ldots, 20$.

2. $y' = \frac{1}{\alpha \cos x} - y \tan x$, $y(0) = 1$, $a = 0$, $b = 1$, $\alpha = 0.2 + 0.01 \cdot k$, $k = 0, 1, 2, \ldots, 20$.

8.10 THE METHOD OF A. N. KRYLOV FOR FINDING THE "INITIAL INTERVAL"

Adams' and Milne's methods call for the existence of four initial values of the required function.

If the right-hand member of the equation

$$y' = f(x, y) \tag{1}$$

is given analytically, then the required initial values can be found by any of the above treated methods (the method of successive approximations, the method of power series, or the Runge-Kutta method). But if the right-hand member of equation (1) is specified in a tabular form, then to construct the "initial interval" it is very convenient to apply the method of successive approach (see [13], [20]), suggested by Krylov and modified by Milne.

Let us introduce the following notation

$$x_i = x_0 + ih, \quad y_i = y(x_i), \quad y'_i = f(x_i, y_i),$$
$$q_i = hy'_i = hf(x_i, y_i) \quad (i = 0, 1, 2, \ldots).$$

The method under consideration is based on iterative processing of points with the aid of the formulas of Euler, Adams, and Milne.

Solving the problem;

1. *First approach.* By Euler's formula we put

$$\Delta y_0^{(1)} = q_0 = hf(x_0, y_0), \quad \Delta y_{-1}^{(1)} = q_0$$

Table 84

Computing the "Initial Interval" by the Method of Successive Approach

Approach No.	i	x	y	Δy	q	Δq	$\Delta^2 q$	$\Delta^3 q$
I	−1	x_{-1}	$y_{-1}^{(1)}$	$\Delta y_{-1}^{(1)}$	$q_{-1}^{(1)}$	$\Delta q_{-1}^{(1)}$	$\Delta^2 q_{-1}^{(1)}$	
	0	x_0	y_0	$\Delta y_0^{(1)}$	q_0	$\Delta q_0^{(1)}$		
	1	x_1	$y_1^{(1)}$		$q_1^{(1)}$			
II	−1	x_{-1}	$y_{-1}^{(2)}$	$\Delta y_{-1}^{(2)}$	$q_{-1}^{(2)}$	$\Delta q_{-1}^{(2)}$	$\Delta^2 q_{-1}^{(2)}$	$\Delta^3 q_{-1}^{(2)}$
	0	x_0	y_0	$\Delta y_0^{(2)}$	q_0	$\Delta q_0^{(2)}$	$\Delta^2 q_0^{(2)}$	
	1	x_1	$y_1^{(2)}$	$\Delta y_1^{(2)}$	$q_1^{(2)}$	$\Delta q_1^{(2)}$		
	2	x_2	$y_2^{(2)}$		$q_2^{(2)}$			
III	0	x_0	y_0	$\Delta y_0^{(3)}$	q_0	$\Delta q_0^{(3)}$	$\Delta^2 q_0^{(3)}$	$\Delta^3 q_0^{(3)}$
	1	x_1	$y_1^{(3)}$	$\Delta y_1^{(3)}$	$q_1^{(3)}$	$\Delta q_1^{(3)}$	$\Delta^2 q_1^{(3)}$	
	2	x_2	$y_2^{(3)}$	$\Delta y_2^{(3)}$	$q_2^{(3)}$	$\Delta q_2^{(3)}$		
	3	x_3	$y_3^{(3)}$		$q_3^{(3)}$			

and compute

$$y_1^{(1)} = y_0 + \Delta y_0^{(1)}, \quad y_{-1}^{(1)} = y_0 - \Delta y_{-1}^{(1)}.$$

Then we find $q_1^{(1)} = hf(x_1, y_1^{(1)})$, $q_{-1}^{(1)} = hf(x, y_{-1}^{(1)})$ and form differences $\Delta q_0^{(1)} = q_1^{(1)} - q_0$, $\Delta q_{-1}^{(1)} = q_0 - q_{-1}^{(1)}$, $\Delta^2 q_{-1}^{(1)} = \Delta q_0^{(1)} - \Delta q_{-1}^{(1)}$. Enter all the results in Section I of Table 84.

II. *Second approach.* Specify the values of Δy_0 and Δy_{-1} by Adam's interpolation formula (3) of Sec. 8.8 and Milne's correction formula (2), rejecting all the differences of order higher than the second

$$\Delta y_0^{(2)} = q_0 + \frac{1}{2}\Delta q_0^{(1)} - \frac{1}{12}\Delta^2 q_{-1}^{(1)}, \quad \Delta y_{-1}^{(2)} = q_0 - \frac{1}{2}\Delta q_{-1}^{(1)} - \frac{1}{12}\Delta^2 q_{-1}^{(1)},$$

so that

$$\Delta y_{-1}^{(2)} + \Delta y_0^{(2)} = y_1 - y_{-1} = 2q_0 + \frac{1}{3}\Delta^2 q_{-1}^{(1)} = \frac{1}{3}(q_{-1}^{(1)} + 4q_0 + q_1^{(1)}).$$

Then we predict the value of $\Delta y_1^{(2)}$ by Adams' extrapolation formula (2) of Sec. 8.8:

$$\Delta y_1^{(2)} = q_1^{(1)} + \frac{1}{2}\Delta q_0^{(1)} + \frac{5}{12}\Delta^2 q_{-1}^{(1)}.$$

This enables us to compute the following approximations:

$$y_{-1}^{(2)} = y_0 - \Delta y_{-1}^{(2)},$$
$$y_1^{(2)} = y_0 + \Delta y_0^{(2)}, \quad y_2^{(2)} = y_1^{(2)} + \Delta y_1^2,$$

and finally we can find

$$q_{-1}^{(2)} = hf(x, y_{-1}^{(2)}), \quad q_1^{(2)} = hf(x_1, y_1^{(2)}), \quad q_2^{(2)} = hf(x_2, y_2^{(2)}).$$

Enter the results in Section II of Table 84 and compute the differences $\Delta q_{-1}^{(2)}$, $\Delta q_0^{(2)}$, $\Delta q_1^{(2)}$, $\Delta^2 q_{-1}^{(2)}$, $\Delta^2 q_0^{(2)}$, $\Delta^3 q_{-1}^{(2)}$.

III. *Third approach.* Now we have a sufficient number of points to continue the computation, but it is necessary to specify the found points by the complete formulas (2) and (3) of Sec. 8.8. This yields

$$\Delta y_0^{(3)} = q_0 + \frac{1}{2}\Delta q_0^{(2)} - \frac{1}{12}\Delta^2 q_{-1}^{(2)} - \frac{1}{24}\Delta^3 q_{-1}^{(2)},$$

$$\Delta y_1^{(3)} = q_1^{(2)} + \frac{1}{2}\Delta q_1^{(2)} - \frac{1}{12}\Delta^2 q_0^{(2)} - \frac{1}{24}\Delta^3 q_{-1}^{(2)}.$$

Now we can also find $\Delta y_2^{(3)}$ by Adams' formula

$$\Delta y_2^{(3)} = q_2^{(2)} + \frac{1}{2}\Delta q_1^{(2)} + \frac{5}{12}\Delta^2 q_0^{(2)} + \frac{3}{8}\Delta^3 q_{-1}^{(2)}.$$

Hence

$$y_1^{(3)} = y_0 + \Delta y_0^{(3)}, \quad y_2^{(3)} = y_1^{(3)} + \Delta y_1^{(3)}, \quad y_3^{(3)} = y_2^{(3)} + \Delta y_2^{(3)}.$$

Then we compute $q_1^{(3)}$, $q_2^{(3)}$, $q_3^{(3)}$, enter the results in Section III of the table and compute the differences $\Delta q_0^{(3)}$, $\Delta q_1^{(3)}$, $\Delta q_2^{(3)}$, and so on.

If spacing h is chosen correctly, then a new calculation by Adams' formulas will not change the value of Δy_0, Δy_1 to a considerable extent. Now the only thing which is left is to specify $y_2^{(3)}$ and to carry out a repeated computation of the values of y_3, q_3, Δq_2, $\Delta^2 q_1$, $\Delta^3 q_0$. Then go on with the computations.

Note. If the right-hand member of equation (1) is given analytically by very cumbersome formulas, then the method of successive approach turns out to be a time saver as compared with the Runge-Kutta method, the extent of accuracy remaining unchanged. As a matter of fact, using the Runge-Kutta method, we have to calculate the values of the function $f(x, y)$ four times for each spacing, and the three initial values are computed twelve times, whereas in computing the "initial interval" by Krylov's method the values of the function $f(x, y)$ are calculated seven times.

Example 1. Find the numerical solution of the equation

$$y' = 2x + y \tag{2}$$

with the initial condition $y(0) = 0.1$, taking $h = 0.1$.

Solution. Let us determine the numerical solution of the given equation by Adams' method, using Krylov's method for computing

Table 85

Solving Equation (2) by the Adams-Krylov Method

Approach No.	i	x	y	Δy	q	Δq	$\Delta^2 q$	$\Delta^3 q$
I	−1	−0.1	0.0900	0.0100	−0.0110	210	0	
	0	0.0	0.1000	0.0100	0.0100	210		
	1	0.1	0.1100		0.0310			
II	−1	−0.1	0.1005	−0.0005	−0.0100	200	20	3
	0	0.0	0.1000	0.0205	0.0100	220	23	
	1	0.1	0.1205	0.0425	0.0320	243		
	2	0.2	0.1630		0.0563			
III	0	0	0.100	0.0208	0.0100	221	23	2
	1	0.1	0.1208	0.0440	0.0321	244	25	4
	2	0.2	0.1648	0.0697	0.0565	269	29	3
	3	0.3	0.2345	0.0980	0.0834	298	32	2
IV	4	0.4	0.3325	0.1295	0.1132	330	34	
	5	0.5	0.4620	0.1641	0.1462	364		
	6	0.6	0.6261		0.1826			

the "initial interval". The results accurate to 10^{-4} are tabulated below (see Table 85).

Filling in the table:

I. *First approach.* Write down in Section I $x_1 = -0.1$, $x_0 = 0$, $x_1 = 0.1$, and compute

$$\Delta y_{-1}^{(1)} = \Delta y_0^{(1)} = 0.1 \cdot 0.1 = 0.0100,$$

$$y_1^{(1)} = y_0 + \Delta y_0^{(1)} = 0.1 + 0.0100 = 0.1100, \quad y_{-1}^{(1)} = 0.1 - 0.0100 = 0.0900,$$

$$q_1^{(1)} = 0.1\,(2x_1 + y_1^{(1)}) = 0.1\,(0.2 + 0.110) = 0.0310,$$

$$q_{-1}^{(1)} = 0.1\,(-0.2 + 0.090) = -0.0110,$$

and also all the differences of the quantities q.

II. *Second approach.*

$$\Delta y_{-1}^{(2)} = 0.0100 - 0.5 \cdot 0.0210 = -0.0005,$$

$$\Delta y_0^{(2)} = 0.0100 + 0.5 \cdot 0.0210 = 0.0205,$$

$$\Delta y_1^{(2)} = 0.0310 + 0.5 \cdot 0.0210 = 0.0425.$$

Then

$$y_{-1}^{(2)} = 0.1000 + 0.0005 = 0.1005, \quad y_1^{(2)} = 0.1000 + 0.0205 = 0.1205,$$

$$y_2^{(2)} = 0.1205 + 0.0425 = 0.1630.$$

Compute

$$q_{-1}^{(2)} = 0.1\,(2x_{-1} + y_{-1}^{(2)}) = 0.1\,(-0.2 + 0.1005) = -0.0100,$$
$$q_1^{(2)} = 0.1\,(2x_1 + y_1^{(2)}) = 0.1\,(0.2 + 0.1205) = 0.0320,$$
$$q_2^{(2)} = 0.1\,(2x_2 + y_2^{(2)}) = 0.1\,(0.4 + 0.1630) = 0.0563.$$

Enter the results in Section II and compute the differences.

III. *Third approach.*

$$\Delta y_0^{(3)} = 0.0100 + \frac{1}{2}\,0.0220 - \frac{1}{12}\,0.0020 = 0.0208,$$

$$\Delta y_1^{(3)} = 0.0320 + \frac{1}{2}\,0.0243 - \frac{1}{12}\,0.0023 = 0.0440,$$

$$\Delta y_2^{(3)} = 0.0563 + \frac{1}{2}\,0.0243 + \frac{5}{12}\,0.0023 + \frac{3}{8}\,0.0003 = 0.0695.$$

Then

$$y_1^{(3)} = 0.1000 + 0.0208 = 0.1208,$$
$$y_2^{(3)} = 0.1208 + 0.0440 = 0.1648,$$
$$y_3^{(3)} = 0.1648 + 0.0695 = 0.2343.$$

The computation of the values of q and their differences shows that the spacing is chosen correctly and the "initial interval" is obtained with a sufficient accuracy (the error not exceeding $3 \cdot 10^{-4}$).

IV. *Continuing the computation.* And this is the end of computing the "initial interval". The further computations are carried out by Adams' method. First specify y_3:

$$\Delta y_2^{\text{corr}} = 0.0565 + \frac{1}{2} \cdot 0.0269 - \frac{1}{12} \cdot 0.0025 = 0.0697,$$

$$y_3^{\text{corr}} = 0.1648 + 0.0697 = 0.2345.$$

Then for $i = 4$ we get

$$\Delta y_3 = q_3 + \frac{1}{2}\Delta q_2 + \frac{5}{12}\Delta^2 q_1 + \frac{3}{8}\Delta^3 q_0 =$$
$$= 0.0834 + \frac{1}{2} \cdot 0.0269 + \frac{5}{12} \cdot 0.0025 + \frac{3}{8} \cdot 0.0002 = 0.0980.$$

Then $y_4 = y_3 + \Delta y_3 = 0.2343 + 0.980 = 0.3323$, the correction leading to the same value.

Calculations for $i = 5$, 6 are carried out analogously. The further computations are recommended to be performed with $h = 0.2$, the found numerical solutions for $x = 0.0$, 0.2, 0.4, 0.6 being used as the "initial interval".

Example 2. On the interval [0; 0.6] find the numerical solution of the system

$$\left.\begin{aligned} \frac{dx}{dt} &= x+y, \\ \frac{dy}{dt} &= -x+y \end{aligned}\right\} \tag{3}$$

with the initial conditions $x(0)=1$, $y(0)=0$ accurate to $5\cdot 10^{-4}$.

Solution. Take $h=0.1$ and introduce the following notation.

$$f_1(x, y)=x+y, \quad f_2(x, y)=-x+y, \quad p=hf_1, \quad q=hf_2.$$

All the values of the "initial intervals" computed by Krylov's method and further computations by Adams' method are contained in Tables 86 and 87.

Filling in the tables:

I. *First approach.* Write down in Tables 86 and 87 the initial values $t_0=0$, $x_0=1$, $y_0=0$ and compute

$$p_0=0.1\cdot 1=0.1, \quad q_0=0.1\,(-1)=-0.1.$$

Using the formulas for the first approach, we get

$$\Delta x_{-1}^{(1)}=\Delta x_0^{(1)}=p_0=0.1, \quad \Delta y_{-1}^{(1)}=\Delta y_0^{(1)}=q_0=-0.1,$$

whence

$$\begin{aligned} x_{-1}^{(1)} &= 1-0.1=0.9, \quad y_{-1}^{(1)}=0.1, \\ x_1^{(1)} &= 1+0.1=1.1, \quad y_1^{(1)}=-0.1. \end{aligned}$$

Then compute

$$\begin{aligned} p_{-1}^{(1)} &= 0.1\,(0.9+0.1)=0.1, \quad q_{-1}^{(1)}=0.1\,(-0.9+0.1)=-0.08, \\ p_1^{(1)} &= 0.1\,(1.1-0.1)=0.1, \quad q_1^{(1)}=0.1\,(-1.1-0.1)=-0.12 \end{aligned}$$

and the corresponding differences.

II. *Second approach.* Using the results of the first approach, compute

$$\Delta x_{-1}^{(2)}=p_0-\frac{1}{2}\Delta p_{-1}-\frac{1}{12}\Delta^2 p_{-1}=0.1,$$

$$\Delta x_0^{(2)}=p_0+\frac{1}{2}\Delta p_0-\frac{1}{12}\Delta^2 p_{-1}=0.1,$$

$$\Delta x_1^{(2)}=p_1+\frac{1}{2}\Delta p_0+\frac{5}{12}\Delta^2 p_{-1}=0.1,$$

$$\Delta y_{-1}^{(2)}=-0.10-\frac{1}{2}(-0.02)=-0.09,$$

$$\Delta y_0^{(2)}=-0.10+\frac{1}{2}(-0.02)=-0.11,$$

$$\Delta y_1^{(2)}=-0.12+\frac{1}{2}(-0.02)=-0.13,$$

Table 86

Solving System (3) by the Adams-Krylov Method. Computing $x(t)$

Approach No.	i	t	x	Δx	$f_1(x, y)$	p	Δp	$\Delta^2 p$	$\Delta^3 p$
I	−1	−0.1	0.9	0.1	1.0	0.10	0	0	
	0	0.0	1.0	0.1	1.0	0.10	0		
	1	0.1	1.1		1.0	0.10			
II	−1	−0.1	0.9	0.1	0.89	0.099	1	−2	0
	0	0.0	1.0	0.1	1.00	0.100	−1	−2	
	1	0.1	1.1	0.1	0.99	0.099	−3		
	2	0.2	1.2		0.96	0.096			
III	0	0.0	1.0000	0.0997	1.0000	0.1000	−11	−23	−5
	1	0.1	1.0997	0.0977	0.9894	0.0989	−34	−28	
	2	0.2	1.1974	0.0939	0.9548	0.0955	−62		
	3	0.3	1.2913		0.8926	0.0893			
IV	0	0.0	1.0000	0.0996	1.0000	0.1000	−11	−24	−4
	1	0.1	1.0996	0.0974	0.9893	0.0989	−35	−28	
	2	0.2	1.1970	0.0927	0.9544	0.0954	−63		−7
	3	0.3	1.2897		0.8908	0.0891		−35	
Correction	3	0.3	1.2895	0.0846			−98		−4
Prediction	4	0.4	1.3741		0.7933	0.0793		−39	
Correction	4	0.4	1.3740	0.0726			−137		−5
Prediction	5	0.5	1.4466		0.6562	0.0656		−44	
Correction	5	0.5	1.4468	0.0570			−181		
Prediction	6	0.6	1.5038		0.4748	0.0475			
Correction	6	0.6	1.5038						

whence

$$x_{-1}^{(2)} = 0.9, \quad y_{-1}^{(2)} = +0.09,$$
$$x_1^{(2)} = 1.1, \quad y_1^{(2)} = -0.11,$$
$$x_2^{(2)} = 1.2, \quad y_2^{(2)} = -0.24.$$

Then compute

$$p_{-1}^{(2)} = 0.1\,(0.9 + 0.09) = 0.099, \quad q_{-1}^{(2)} = 0.1\,(-0.9 + 0.09) = -0.081,$$
$$p_1^{(2)} = 0.1\,(1.1 - 0.11) = 0.099, \quad q_1^{(2)} = 0.1\,(-1.1 - 0.11) = -0.121,$$
$$p_2^{(2)} = 0.1\,(1.2 - 0.24) = 0.096, \quad q_2^{(2)} = 0.1\,(-1.2 - 0.24) = -0.144$$

and the corresponding differences.

Table 87

Solving System (3) by the Adams-Krylov Method. Computing $y(t)$

Approach No.	i	t	y	Δy	$f_2(x, y)$	q	Δq	$\Delta^2 q$	$\Delta^3 q$
I	−1	−0.1	0.1	−0.1	−0.8	−0.08	−2	0	
	0	0.0	0.0	−0.1	−1.0	−0.10	−2		
	1	0.1	−0.1		−1.2	−0.12			
II	−1	−0.1	0.09	−0.09	−0.81	−0.081	−19	−2	0
	0	0.0	0.00	−0.11	−1.00	−0.100	−21	−2	
	1	0.1	−0.11	−0.13	−1.21	−0.121	−23		
	2	0.2	−0.24		−1 44	−0.144			
III	0	0.0	0.0000	−0.1103	−1.0000	−0.1000	−210	−20	0
	1	0.1	−0.1103	−0.1323	−1.2100	−0.1210	−230	−20	0
	2	0.2	−0.2426	−0.1561	−1.4400	−0.1440	−250	−20	
	3	0.3	−0.3987		−1.6900	−0.1690			
IV	0	0.0	0.0000	−0.1103	−1.0000	−0.1000	−210	−20	1
	1	0.1	−0.1103	−0.1323	−1.2099	−0.1210	−230	−19	
	2	0.2	−0.2426	−0.1563	−1.4396	−0.1440	−249		2
	3	0.3	−0.3989		−1.6886	−0.1689		−17	
Correction	3	0.3	−0.3989	−0.1819			−266		1
Prediction	4	0.4	−0.5808		−1.9549	−0.1955		−16	
Correction	4	0.4	−0.5810	−0.2094			−282		2
Prediction	5	0.5	−0.7904		−2.2370	−0.2237		−14	
Correction	5	0.5	−0.7905	−0.2385			−296		
Prediction	6	0.6	−1.0290		−2.5328	−0.2533			
Correction	6	0.6	−1.0289						

III. *Third approach.* Using the results of the second approach, compute

$$\Delta x_0^{(3)} = p_0 + \frac{1}{2}\Delta p_0 - \frac{1}{12}\Delta^2 p_{-1} - \frac{1}{24}\Delta^3 p_{-1} = 0.0997,$$

$$\Delta x_1^{(3)} = p_1 + \frac{1}{2}\Delta p_1 - \frac{1}{12}\Delta^2 p_0 - \frac{1}{24}\Delta^3 p_{-1} = 0.0977,$$

$$\Delta x_2^{(3)} = p_2 + \frac{1}{2}\Delta p_1 + \frac{5}{12}\Delta^2 p_0 + \frac{3}{8}\Delta^3 p_{-1} = 0.0939,$$

$$\Delta y_0^{(3)} = -0.100 + \frac{1}{2}(-0.021) - \frac{1}{12}(-0.002) = -0.1103,$$

$$\Delta y_1^{(3)} = -0.121 + \frac{1}{2}(-0.023) - \frac{1}{12}(-0.002) = -0.1323,$$

$$\Delta y_2^{(3)} = -0.144 + \frac{1}{2}(-0.023) + \frac{5}{12}(-0.002) = -0.1561,$$

whence

$$x_1^{(3)} = 1.0997, \quad y_1^{(3)} = -0.1103,$$
$$x_2^{(3)} = 1.1974, \quad y_2^{(3)} = -0.2426,$$
$$x_3^{(3)} = 1.2913, \quad y_3^{(3)} = -0.3987.$$

Then compute

$$p_1^{(3)} = 0.1\,(1.0997 - 0.1103) = 0.0989, \quad q_1^{(3)} = -0.1210,$$
$$p_2^{(3)} = 0.1\,(1.1974 - 0.2426) = 0.0955, \quad q_2^{(3)} = -0.1440,$$
$$p_3^{(3)} = 0.1\,(1.2913 - 0.3987) = 0.0893, \quad q_3^{(3)} = -0.1690$$

and the appropriate differences.

IV. *Fourth approach* is carried out for the checking purpose. We get

$$\Delta x_0^{(4)} = p_0 + \frac{1}{2}\Delta p_0 - \frac{1}{12}\Delta^2 p_0 + \frac{1}{24}\Delta^3 p_0 = 0.0996,$$

$$\Delta x_1^{(4)} = p_1 + \frac{1}{2}\Delta p_1 - \frac{1}{12}\Delta^2 p_0 - \frac{1}{24}\Delta^3 p_0 = 0.0974,$$

$$\Delta x_2^{(4)} = p_2 + \frac{1}{2}\Delta p_2 - \frac{1}{12}\Delta^2 p_1 - \frac{1}{24}\Delta^3 p_0 = 0.0927,$$

$$\Delta y_0^{(4)} = -0.1000 + \frac{1}{2}(-0.0210) - \frac{1}{12}(-0.0020) = -0.1103.$$

$$\Delta y_1^{(4)} = -0.1210 + \frac{1}{2}(-0.0230) - \frac{1}{12}(-0.0020) = -0.1323,$$

$$\Delta y_2^{(4)} = -0.1440 + \frac{1}{2}(-0.0250) - \frac{1}{12}(-0.0020) = -0.1563,$$

wherefrom

$$x_1^{(4)} = 1.0996, \quad y_1^{(4)} = -0.1103,$$
$$x_2^{(4)} = 1.1970, \quad y_2^{(4)} = -0.2426,$$
$$x_3^{(4)} = 1.2897, \quad y_3^{(4)} = -0.3989.$$

Then compute

$$p_1^{(4)} = 0.1\,(1.0996 - 0.1103) = 0.0989, \quad q_1^{(4)} = -0.1210,$$
$$p_2^{(4)} = 0.1\,(1.1970 - 0.2426) = 0.0954, \quad q_2^{(4)} = -0.1440,$$
$$p_3^{(4)} = 0.1\,(1.2897 - 0.3989) = 0.0891, \quad q_3^{(4)} = -0.1689$$

and the corresponding differences.

Since the results of the third and fourth approaches differ only by permissible quantities, we take the results of the fourth approach for the "initial interval" and carry out the further computations by Adams' method, using the following as the prediction

formula

$$\Delta x_i^{\text{pred.}} = p_i + \frac{1}{2}\Delta p_{i-1} + \frac{5}{12}\Delta^2 p_{i-2} + \frac{3}{8}\Delta^3 p_{i-3},$$

and the correction formula

$$\Delta x_i^{\text{corr.}} = p_i + \frac{1}{2}\Delta p_i - \frac{1}{12}\Delta^2 p_{i-1} - \frac{1}{24}\Delta^3 p_{i-2}.$$

For example, first correct x_3:

$$\Delta x_2 = p_2 + \frac{1}{2}\Delta p_2 - \frac{1}{12}\Delta^2 p_1 - \frac{1}{24}\Delta^3 p_0 =$$

$$= 0.0954 + \frac{1}{2}(-0.0063) - \frac{1}{12}(-0.0028) - \frac{1}{24}(-0.0004) = 0.0925$$

$$x_3^{\text{corr.}} = x_2 + \Delta x_2 = 1.2895.$$

Then predict x_4:

$$\Delta x_3 = p_3 + \frac{1}{2}\Delta p_2 + \frac{5}{12}\Delta^2 p_1 + \frac{3}{8}\Delta^3 p_3 =$$

$$= 0.0891 + \frac{1}{2}(-0.0063) + \frac{5}{12}(-0.0028) + \frac{3}{8}(-0.0004) = 0.0846,$$

$$x_4^{\text{pred.}} = x_3 + \Delta x_3 = 1.3741.$$

Note. Applying the Seidel method used for iterative solution of systems of linear equations (Chapter 3, Sec. 3.10), we can sometimes accelerate the process of approach in the Krylov method. In this case the Seidel method consists in the following. Beginning with the second approach, we make use of the computed values of p_i, Δp_{i-1}, ... for computing each subsequent difference Δx_i $(i > 0)$.

In the above considered example the use of the Seidel method makes it possible to carry out the computations with a doubled spacing $h = 0.2$. The four approaches computed by the Krylov method and accelerated by the Seidel method are presented in Table 88 and 89.

Filling in the tables:

I. *The first approach* is computed in the same way as above but with the spacing $h = 0.2$.

II. *The second approach* is accomplished beginning with the point $t = 0$. Using the results of the first approach, we first compute

$$\Delta x_0^{(2)} = p_0 + \frac{1}{2}\Delta p_0 - \frac{1}{12}\Delta^2 p_{-1} = 0.20,$$

$$\Delta y_0^{(2)} = q_0 + \frac{1}{2}\Delta q_0 - \frac{1}{12}\Delta^2 q_{-1} = -0.20 + \frac{1}{2}(-0.08) = -0.24,$$

which yields $x_1^{(2)} = 1.2$, $y_1^{(2)} = -0.24$. Then compute

$p_1^{(2)} = 0.2\,(1.20 - 0.24) = 0.192$, $q_1^{(2)} = 0.2\,(-1.20 - 0.24) = -0.288$

and the corresponding differences $\Delta p_0 = -0.008$, $\Delta q_0 = -0.088$.

Table 88

Solving System (3) by the Krylov-Seidel Method. Computing $x(t)$

Approach No.	i	t	x	Δx	$f_1(x, y)$	p	Δp	$\Delta^2 p$	$\Delta^3 p$
I	−1	−0.2	0.8	0.2	1.0	0.20	0	0	
	0	0.0	1.0	0.2	1.0	0.20	0		
	1	0.2	1.2		1.0	0.20			
II	0	0.0	1.000	0.200	1.000	0.2000	−80	−208	
	1	0.2	1.200	0.188	0.960	0.1920	−288		
	2	0.4	1.388		0.816	0.1632			
III	0	0.0	1.0000	0.1977	1.0000	0.2000	−90	−232	−79
	1	0.2	1.1977	0.1778	0.9550	0.1910	−322	−311	
	2	0.4	1.3755	0.1330	0.7940	0.1588	−633		
	3	0.6	1.5085		0.4776	0.0955			
IV	0	0.0	1.0000	0.1971	1.0000	0.2000	−91	−232	−82
	1	0.2	1.1971	0.1770	0.9544	0.1909	−323	−314	
	2	0.4	1.3741	0.1299	0.7930	0.1586	−637		
	3	0.6	1.5040		0.4747	0.0949			
Correction	3	0.6	1.5038		0.4749	0.0950			

And only after this we calculate the differences $\Delta x_1^{(2)}$ and $\Delta y_1^{(2)}$, using the values of p_1 and q_1 from the second approach:

$$\Delta x_1^{(2)} = p_1 + \frac{1}{2}\Delta p_0 \frac{5}{12}\Delta^2 p_{-1} = 0.192 + \frac{1}{2}(-0.008) = 0.188,$$

$$\Delta y_1^{(2)} = -0.288 + \frac{1}{2}(-0.088) = -0.332.$$

Then compute

$$p_2^{(2)} = 0.2\,(1.388 - 0.572) = 0.1632, \quad q_2^{(2)} = 0.2\,(-1.388 - 0.572) = -0.3920$$

and the corresponding differences.

III. *Third approach.* Using the results of the second approach, find only the first point:

$$\Delta x_0^{(3)} = p_0 + \frac{1}{2}\Delta p_0 - \frac{1}{12}\Delta^2 p_0 = 0.2000 + \frac{1}{2}(-0.0080) - \frac{1}{12}(-0.0208) = 0.1977,$$

$$\Delta y_0^{(3)} = -0.2000 + \frac{1}{2}(-0.0880) - \frac{1}{12}(-0.0160) = -0.2427,$$

$$x_1^{(3)} = 1.1977, \quad y_1^{(3)} = -0.2427.$$

Table 89

Solving System (3) by the Krylov-Seidel Method. Computing $y(t)$

Approach No.	i	t	y	Δy	$f_2(x, y)$	q	Δq	$\Delta^2 q$	$\Delta^3 q$
I	—1	—0.2	0.2	—0.2	—0.6	—0.12	—8	0	
	0	0.0	0.0	—0.2	—1.0	—0.20	—8		
	1	0.2	—0.2		—1.4	—0.28			
II	0	0.0	0.000	—0.240	—1.000	—0.2000	—880	—160	
	1	0.2	—0.240	—0.332	—1.440	—0.2880	—1040		
	2	0.4	—0.572		—1.960	—0.3920			
III	0	0.0	0.0000	—0.2427	—1.0000	—0.2000	—881	—152	20
	1	0.2	—0.2427	—0.3388	—1.4404	—0.2881	—1033	—132	
	2	0.4	—0.5815	—0.4494	—1.9570	—0.3914	—1165		
	3	0.6	—1.0309		—2.5394	—0.5079			
IV	0	0.0	0.0000	—0.2427	—1.0000	—0.2000	—880	—150	23
	1	0.2	—0.2427	—0.3384	—1.4398	—0.2880	—1030	—127	
	2	0.4	—0.5811	—0.4482	—1.9552	—0.3910	—1157		
	3	0.6	—1.0293		—2.5333	—0.5067			
Correction	3	0.6	—1.0289		—2.5327	—0.5065			

Then compute

$$p_1^{(3)} = 0.2\,(1.1977 - 0.2427) = 0.1910,$$
$$q_1^{(3)} = 0.2\,(-1.1977 - 0.2427) = -0.2881$$

and the corresponding differences.

Then find the second point, using the values of p_1 and q_1 from the third approach (and the second differences from the second approach):

$$\Delta x_1^{(3)} = 0.1910 + \frac{1}{2}(-0.0090) + \frac{5}{12}(-0.0208) = 0.1778,$$

$$\Delta y_1^{(3)} = -0.2881 + \frac{1}{2}(-0.0881) + \frac{5}{12}(-0.0160) = -0.3388,$$

$$x_2^{(3)} = 1.3755, \quad y_2^{(3)} = -0.5815.$$

Again compute

$$p_2^{(3)} = 0.2\,(1.3755 - 0.5815) = 0.1588,$$
$$q_2^{(3)} = 0.2\,(-1.3755 - 0.5815) = -0.3914$$

and the corresponding differences.

And finally find

$$\Delta x_2^{(3)} = 0.1588 + \frac{1}{2}(-0.0322) + \frac{5}{12}(-0.0232) = 0.1330,$$

$$\Delta y_2^{(3)} = -0.3914 + \frac{1}{2}(-0.1033) + \frac{5}{12}(-0.0152) = -0.4494,$$

$$x_3^{(3)} = 1.5085, \quad y_3^{(3)} = -1.0309.$$

Then compute p_3, q_3 and the corresponding differences.

IV. *Fourth approach.* The first point is computed in a similar way, taking into account $\Delta^3 p_0$ and $\Delta^3 q_0$, respectively

$$\Delta x_0^{(4)} = p_0 + \frac{1}{2}\Delta p_0 - \frac{1}{12}\Delta^2 p_0 + \frac{1}{24}\Delta^3 p_0 = 0.2000 - 0.0045 - 0.0019 - 0.0003 = 0.1971,$$

$$\Delta y_0^{(4)} = -0.2000 - 0.0441 + 0.0013 + 0.0001 = -0.2427,$$

$$x_1^{(4)} = 1.1971, \quad y_1^{(4)} = -0.2427.$$

Now make use of the values of p_1 and q_1 for computing the second point:

$$\Delta x_1^{(4)} = p_1 + \frac{1}{2}\Delta p_1 - \frac{1}{12}\Delta^2 p_0 - \frac{1}{24}\Delta^3 p_0 = 0.1909 - 0.0161 + 0.0019 + 0.0003 = 0.1770,$$

$$\Delta y_1^{(4)} = -0.2880 - 0.0516 + 0.0013 - 0.0001 = -0.3384,$$

$$x_2^{(4)} = 1.3741, \quad y_2^{(4)} = -0.5811.$$

Then compute p_2 and q_2 and use them for computing the third point:

$$\Delta x_2^{(4)} = 0.1586 - 0.0316 + 0.0026 + 0.0003 = 0.1299,$$

$$\Delta y_2^{(4)} = -0.3910 - 0.0582 + 0.0011 - 0.0001 = -0.4482,$$

$$x_3^{(4)} = 1.5040, \quad x_3^{(4)} = -1.0293.$$

Since the difference between the third and fourth approaches is small enough, we stop further computations to correct the last obtained point by Adams' correction formula. This yields

$$\Delta x_2 = 0.1586 - 0.0318 + 0.0026 + 0.0003 = 0.1297, \quad x_3 = 1.5038,$$
$$\Delta y_2 = -0.3910 - 0.0578 + 0.0011 - 0.0001 = -0.4478, \quad y_3 = -1.0289.$$

Problems

Using the method of successive approach and Adams' method, find the approximate solutions of the differential equations with spacing $h = 0.05$ on the interval $[0, c]$:

1. $y' = \cos^2(ax + y) + by$, $y(0) = 0$, $c = 0.25$, $a = 1.0 + 0.4 \cdot n$, $n = 0, 1, \ldots, 5$, $b = 1.0 + 0.4 \cdot k$, $k = 0, 1, \ldots, 5$.
2. $y' = e^{-(a+xy)} + by$, $y(0) = 0$, $c = 0.25$, $a = 1.0 + 0.4 \cdot n$, $n = 0, 1, \ldots, 5$, $b = 1.0 + 0.4 \cdot k$, $k = 0, 1, \ldots, 5$.
3. $y' = -axy$, $y(0) = 1$, $c = 0.2$, $a_1 = 1.8$, $a_2 = 2.0$.
4. $y' = -\frac{a(y+x)^k}{a(y+x)^k + 1}$, $y(0) = 0$, $c = 0.2$, $a = 1.0 + 0.3 \cdot n$, $n = 0, 1, 2$, $k = 1, 2, 3, 4$.

9

BOUNDARY VALUE PROBLEMS FOR ORDINARY DIFFERENTIAL EQUATIONS

9.1 THE STATEMENT OF PROBLEMS

Let it be given a differential equation of the second order

$$F(x, y, y', y'') = 0. \tag{1}$$

A two-point boundary problem for equation (1) is set in the following way: find the function $y = y(x)$, which inside the interval $[a, b]$ satisfies the function (1), and at the endpoints of the interval, the boundary conditions

$$\left.\begin{aligned} \varphi_1[y(a), \quad y'(a)] = 0, \\ \varphi_2[y(b), \quad y'(b)] = 0. \end{aligned}\right\} \tag{2}$$

Consider the case when equation (1) and boundary conditions (2) are linear. Such a boundary value problem is called the *linear boundary value problem*. In this case the differential equation and boundary conditions are written as follows:

$$y'' + p(x)\, y' + q(x)\, y = f(x), \tag{3}$$

$$\left.\begin{aligned} \alpha_0 y(a) + \alpha_1 y'(a) = A, \\ \beta_0 y(b) + \beta_1 y'(b) = B, \end{aligned}\right\} \tag{4}$$

where $p(x)$, $q(x)$, $f(x)$ are known quantities continuous on the interval $[a, b]$ of the function; α_0, α_1, β_0, β_1, A, B are given constants, and $|\alpha_0| + |\alpha_1| \neq 0$, and $|\beta_0| + |\beta_1| \neq 0$.

If $A = B = 0$, then the boundary conditions (4) are called *uniform*. The methods of approximate solutions of the boundary value problems are distinguished between the *difference methods* and *analytical methods*. These methods are considered in the subsequent sections.

9.2 THE METHOD OF FINITE DIFFERENCES FOR SECOND-ORDER LINEAR DIFFERENTIAL EQUATIONS

Let $x_0=a$, $x_n=b$, $x_i=x_0+ih$ $(i=1, 2, \ldots, n-1)$ be a system of equally spaced points with spacing $h=\frac{b-a}{n}$ and

$$p_i=p(x_i), \quad q_i=q(x_i), \quad f_i=f(x_i).$$

Let us denote the obtained approximate values of the required function $y(x)$ and its derivatives $y'(x)$, $y''(x)$ at point x_i by y_i, y_i', y_i'', respectively. Let us replace approximately the derivatives $y'(x_i)$, $y''(x_i)$ at each inside point by the following finite-difference relations

$$y_i'=\frac{y_{i+1}-y_{i-1}}{2h}, \quad y_i''=\frac{y_{i+1}-2y_i+y_{i-1}}{h^2}. \tag{1}$$

Then we get the system

$$\left.\begin{aligned} &\frac{y_{i+1}-2y_i+y_{i-1}}{h^2}+p_i\frac{y_{i+1}-y_{i-1}}{2h}+q_iy_i=f_i \quad (i=1, 2, \ldots, n-1), \\ &\alpha_0y_0+\alpha_1\frac{y_1-y_0}{h}=A, \quad \beta_0y_n+\beta_1\frac{y_n-y_{n-1}}{h}=B. \end{aligned}\right\} \tag{2}$$

For a large n the direct solution of system (2) becomes rather cumbersome. Section 9.3 introduces a simple method developed specially for solving systems of such kind. The error estimate of the method of finite differences for Problem (3), (4) of Sec. 9.1 has the form

$$|y_i-y(x_i)|\leqslant\frac{h^2M_4}{96}(b-a)^2, \tag{3}$$

where $y(x_i)$ is the exact solution for $x=x_i$, $M_4=\max\limits_{[a,\ b]}|y^{(4)}(x)|$.

The accuracy of the difference method can be considerably increased by using multipoint difference schemes when replacing derivatives (see [39]).

In practical problems we often come across equations, in which the functions $p(x)$, $q(x)$, $f(x)$ are given by tables with a certain spacing h. It is only natural to solve such equations by the difference method with a given spacing h (see Example 4 of Sec. 9.3).

Example 1. Using the method of finite differences, find the solution of the boundary value problem

$$\left.\begin{aligned} &x^2y''+xy'=1, \\ &y(1)=0, \quad y(1.4)=\frac{1}{2}\ln^2(1.4)=0.0566. \end{aligned}\right\} \tag{4}$$

Solution. Using formula (1), replace equation (4) by the system

of finite-difference equation

$$x_i^2 \frac{y_{i+1}-2y_i+y_{i-1}}{h^2} + x_i \frac{y_{i+1}-y_{i-1}}{2h} = 1.$$

Collecting the like terms, we get

$$y_{i-1}(2x_i^2 - hx_i) - 4x_i^2 y_i + y_{i+1}(2x_i^2 + hx_i) = 2h^2. \tag{5}$$

Choosing spacing h equal to 0.1, we obtain three interval points $x_i = 0.1 \cdot i + 1$ $(i = 1, 2, 3)$. Writing equation (5) for each of these points, we get the system

$$\left.\begin{aligned} 2.31y_0 - 4.84y_1 + 2.53y_2 &= 0.02, \\ 2.76y_1 - 5.76y_2 + 3.00y_3 &= 0.02, \\ 3.25y_2 - 6.76y_3 + 3.51y_4 &= 0.02. \end{aligned}\right\} \tag{6}$$

At the boundary points we have

$$y_0 = 0, \quad y_4 = 0.0566.$$

Making use of these values, solve system (6) to obtain

$$y_1 = 0.0046, \quad y_2 = 0.0167, \quad y_3 = 0.0345.$$

For the sake of comparison we present the values of the exact solution of equation (7) $y = \frac{1}{2}\ln^2 x$ for the corresponding points $y(x_1) = 0.0047$, $y(x_2) = 0.0166$, $y(x_3) = 0.0344$.

9.3 THE "PASSAGE" METHOD

Consider the "passage" method for solving the system which is obtained when equation (3) and the second boundary condition (4) (see Sec. 9.1) are replaced by the *central finite-difference relations*

$$\left.\begin{aligned} &\frac{y_{i+1}-2y_i+y_{i-1}}{h^2} + p_i \frac{y_{i+1}-y_{i-1}}{2h} + q_i y_i = f_i \quad (i = 1, 2, \ldots, n-1), \\ &\alpha_0 y_0 + \alpha_1 \frac{y_1 - y_0}{h} = A, \quad \beta_0 y_n + \beta_1 \frac{y_{n+1} - y_{n-1}}{2h} = B. \end{aligned}\right\} \tag{1}$$

Write the first $n-1$ equations of system (1) in the form

$$y_{i+1} + m_i y_i + k_i y_{i-1} = \frac{2h^2 f_i}{2 + hp_i} = \varphi_i,$$

where

$$m_i = \frac{2q_i h^2 - 4}{2 + hp_i}, \quad k_i = \frac{2 - hp_i}{2 + hp_i}. \tag{2}$$

Then reduce these equations to the form

$$y_i = c_i(d_i - y_{i+1}) \quad (i = 1, 2, \ldots, n-1), \tag{3}$$

where the coefficients c_i, d_i are computed by the following formulas: for $i=1$

$$\left.\begin{aligned} c_1 &= \frac{\alpha_1-\alpha_0 h}{m_1(\alpha_1-\alpha_0 h)+k_1\alpha_1}, \\ d_1 &= \frac{2f_1h^2}{2+p_1h}+k_1\frac{Ah}{\alpha_1-\alpha_0 h}=\varphi_1+k_1\frac{Ah}{\alpha_1-\alpha_0 h}; \end{aligned}\right\} \tag{4}$$

for $i=2, 3, \ldots, n$

$$c_i=\frac{1}{m_i-k_ic_{i-1}}, \quad d_i=\frac{2f_ih^2}{2+hp_i}-k_ic_{i-1}d_{i-1}=\varphi_i-k_ic_i-1d_{i-1}. \tag{5}$$

The computations are carried out in the following succession:

Direct procedure. Find m_i, k_i by formulas (2). Compute c_1, d_1, and then, using reccurrence formulas (5), successively find c_i, d_i $(i=2, \ldots, n)$.

Reverse procedure. Write down the equation (3) for $i=n$, $i=n-1$ and the last equation of system (1):

$$\left.\begin{aligned} &y_n=c_n(d_n-y_{n+1}), \\ &y_{n-1}=c_{n-1}(d_{n-1}-y_n), \\ &\beta_0y_n+\beta_1\frac{y_{n+1}-y_{n-1}}{2h}=B. \end{aligned}\right\} \tag{6}$$

Solving this system with respect to y_n, we have

$$y_n=\frac{2Bh-\beta_1(d_n-c_{n-1}d_{n-1})}{2\beta_0h+\beta_1\left(c_{n-1}-\frac{1}{c_n}\right)}. \tag{7}$$

Using the known numbers c_n, d_n, c_{n-1}, d_{n-1}, find y_n. The values of y_i $(i=n-1, \ldots, 1)$ are obtained from the recurrence formulas (3), y_0 being computed from the equation $\alpha_0y_0+\alpha_1\frac{y_1-y_0}{h}=A$ (1):

$$y_0=\frac{\alpha_1y_1-Ah}{\alpha_1-\alpha_0h}. \tag{8}$$

Example 1. Using the "passage" method, find the approximate solution of the equation

$$y''-2xy'-2y=-4x, \tag{9}$$

satisfying the boundary conditions

$$y(0)-y'(0)=0, \quad 2y(1)-y'(1)=1. \tag{10}$$

Solution. Put $h=0.1$ and replace equation (9) and boundary conditions (10) by a system of finite-difference equations

$$\frac{y_{i+1}-2y_i+y_{i-1}}{h^2}-2x_i\frac{y_{i+1}-y_{i-1}}{2h}-2y_i=-4x_i \ (i=1, 2, \ldots, 9),$$

$$y_0-\frac{y_1-y_0}{h}=0, \quad 2y_{10}-\frac{y_{11}-y_0}{2h}=1.$$

Collecting the like terms, we get

$$y_{i+1} - \frac{2+2h^2}{1-x_ih}\,y_i + \frac{1+x_ih}{1-x_ih}\,y_{i-1} = -\frac{4h^2}{1-x_ih}\,x_i.$$

Thus, we have

$$m_i = -\frac{2+2h^2}{1-x_ih},\quad k_i = \frac{1+x_ih}{1-x_ih},\quad \varphi_i = -\frac{4h^2}{1-x_ih}\,x_i \ (i=0,\ 1,\ 2,\ \ldots,\ 10),$$
$$\alpha_0 = 1,\ \alpha_1 = -1,\ \beta_0 = 2,\ \beta_1 = -1,\ A = 0,\ B = 1.$$

Direct procedure. Enter the numbers x_i, $0.1i$ in Table 90 and compute the values of m_i, k_i, φ_i. Then, by formulas (4), we find

$$c_1 = \frac{-1.1}{2.040\cdot 1.1 - 1.020} = -0.899,\quad d_1 = -0.004.$$

Write the obtained numbers in Table 90 and proceed with successive computation of c_i and d_i by formulas (5). Thus, for $i=2$ we get

$$c_2 = \frac{1}{m_2 - k_2c_1} = \frac{1}{-2.060 + 1.040\cdot 0.899} = -0.889,$$
$$d_2 = \varphi_2 - k_2c_1d_1 = -0.008 - 1.040\cdot 0.899\ (-0.004) = -0.012.$$

Computations for $i=3,\ 4,\ \ldots,\ 10$ are carried out analogously. All the results are entered in the columns for the direct procedure of Table 90.

Reverse procedure. Determine y_{10} by formula (7):

$$y_{10} = \frac{0.2 - 0.222 - 0.810\cdot 0.180}{0.4 + 0.810 - \dfrac{1}{0.797}} = 3.73$$

Table 90

The "Passage" Method for Problem (9), (10)

i	x_i	m_i	k_i	φ_i	Direct procedure		Reverse procedure	$y(x_i)$
					c_i	d_i	y_i	
0	0.0						1.03	1.00
1	0.1	−2.040	1.020	−0.004	−0.899	−0.004	1.13	1.11
2	0.2	−2.061	1.040	−0.008	−0.889	−0.012	1.26	1.24
3	0.3	−2.083	1.062	−0.012	−0.878	−0.023	1.41	1.39
4	0.4	−2.105	1.083	−0.017	−0.868	−0.039	1.60	1.57
5	0.5	−2.127	1.105	−0.021	−0.856	−0.058	1.81	1.78
6	0.6	−2.149	1.128	−0.025	−0.845	−0.081	2.06	2.03
7	0.7	−2.172	1.151	−0.030	−0.833	−0.109	2.36	2.33
8	0.8	−2.196	1.174	−0.035	−0.822	−0.142	2.72	2.70
9	0.9	−2.220	1.198	−0.040	−0.810	−0.180	3.17	3.15
10	1.0	−2.244	1.222	−0.044	−0.797	−0.222	3.73	3.72

Write down the obtained value in the last row of Table 90 for $i=10$. Then successively find y_i ($i=9, 8, \ldots, 1$) by formulas (3):

$$y_9 = c_9(d_9 - y_{10}) = -0.810(-0.18 - 3.73) = 3.17,$$

$$y_8 = c_8(d_8 - y_9) = 0.822(-0.14 - 3.17) = 2.72 \text{ and so on.}$$

And finally, by formula (8), we get

$$y_0 = \frac{-1.13}{-1.1} = 1.03.$$

For the sake of comparison the last column of Table 90 gives the corresponding values of the exact solution of Problem (9), (10) $y = x + e^{x^2}$.

Note. In Sec. 9.2 formula (3) was suggested for estimating the error of the method of finite differences. But it is practically of little use. It is standard practice to use double computation (see Example 2), and, proceeding from the Runge principle (see [18]), to obtain an approximate estimate of the error of the value of y_i^*:

$$|y_i^* - y(x_i)| \approx \frac{1}{3}|y_i^* - y_i|,$$

where $y(x_i)$ is the exact solution of the boundary value problem for the point $x = x_i$, and y_i, y_i^* are the values of approximate solutions for the same point obtained with the spacings h and $\frac{h}{2}$, respectively.

Example 2. Using the "passage" method, find to within 0.001 the solution of the equation

$$y'' + 2xy' + 2y = 4x, \tag{11}$$

satisfying the boundary conditions

$$y(0) = 1, \quad y(0.5) = e^{-0.25} + 0.5 = 1.279. \tag{12}$$

Solution. To find the approximate solution of equation (11) with the preassigned accuracy, first carry out the computations with the spacing $h = 0.05$, then with $h = 0.1$, and compare the results obtained.

For the given equation we have

$$p(x) = 2x, \quad q(x) = 2, \quad f(x) = 4x. \tag{13}$$

Using formulas (2), form a table of the values of m_i, k_i, $\frac{2f_ih^2}{2 + hp_i} = \varphi_i$ with $h = 0.05$. Taking into account that here we have $\alpha_0 = 1$, $\alpha_1 = 0$, $\beta_0 = 1$, $\beta_1 = 0$, find c_1, d_1 by formulas (4):

$$c_1 = \frac{1}{m_1} = -0.502, \quad d_1 = \varphi_1 - k_1 = -0.994.$$

Applying successively formulas (5), filling the table of values of c_i, d_i ($i = 2, \ldots, 9$).

From the boundary condition we write down $y_{10} = y(0.5) = 1.279$ and compute y_i $(i = 9, 8, \ldots, 1)$, using the reccurrence formula (3). The results of the computations are entered in Table 91.

Now take the spacing $h = 0.1$, form a table of the values of m_i, k_i, $\frac{2f_ih^2}{2+hp_i} = \varphi_i$ with the new spacing and compute c_i, d_i. For instance, for c_1, d_1 we get

$$c_1 = \frac{1}{m_1} = -0.510, \quad d_1 = \frac{2f_1h^2}{2+hp_1} - k_1 = -0.976.$$

Enter the obtained numbers in Table 92, and, using formula (3), compute successively the values of y_i $(i = 4, 3, 2, 1)$.

Comparing the corresponding values of y_i for $x = 0.1$; 0.2; 0.3; 0.4 contained in Tables 91 and 92, we see that their difference

Table 91

The "Passage" Method for Problem (11), (12) with $h = 0.05$

i	x_i	m_i	k_i	$\frac{2f_ih^2}{2+hp_i} = \varphi_i$	Direct procedure		Reverse procedure
					c_i	d_i	y_i
1	0.05	—1.990	0.995	0.0005	—0.502	—0.994	1.046
2	0.10	—1.985	0.990	0.0010	—0.672	—0.493	1.088
3	0.15	—1.980	0.985	0.0015	—0.759	—0.325	1.125
4	0.20	—1.975	0.980	0.0020	—0.812	—0.239	1.158
5	0.25	—1.970	0.975	0.0025	—0.848	—0.187	1.187
6	0.30	—1.965	0.970	0.0030	—0.876	—0.151	1.213
7	0.35	—1.967	0.966	0.0035	—0.897	—0.124	1.234
8	0.40	—1.956	0.961	0.0040	—0.914	—0.103	1.253
9	0.45	—1.951	0.956	0.0045	—0.928	—0.087	1.267
10	0.50						1.279

Table 92

The "Passage" Method for Problem (11), (12) with $h = 0.1$

i	x_i	m_i	k_i	$\frac{2f_ih^2}{2+hp_i} = \varphi_i$	Direct procedure		Reverse procedure
					c_i	d_i	y_i
1	0.1	—1.960	0.980	0.004	—0.510	—0.976	1.089
2	0.2	—1.941	0.961	0.008	—0.689	—0.470	1.160
3	0.3	—1.992	0.942	0.012	—0.786	—0.293	1.214
4	0.4	—1.904	0.923	0.015	—0.848	—0.197	1.252
5	0.5						1.279

does not exceed 0.002, and, hence, the error of the more accurate solution with $h=0.05$ does not exceed $\frac{0.002}{3}$. Thus, the value of y_i found with the spacing $h=0.05$ satisfy the condition of the problem and may be taken for the solution.

Example 3. Find the solution of the equation

$$y''+p(x)y'+q(x)y=f(x), \tag{14}$$

satisfying the boundary conditions

$$y(0)=0, \quad y(0.4)=0.335, \tag{15}$$

if the functions $p(x)$, $q(x)$, $f(x)$ are given in a tabular form (Table 93).

Solution. The results of computations with spacing $h=0.1$ are listed in Table 94.

Table 93

Coefficients of Equation (14)

x_i	p_i	q_i	f_i
0	—1.20	0.78	0.28
0.1	—1.22	0.75	0.25
0.2	—1.24	0.69	0.20
0.3	—1.26	0.64	0.18
0.4	—1.28	0.60	0.15

Table 94

The "Passage" Method for Problem (14), (15)

x_i	m_i	k_i	h^2f_i	c_i	d_i	y_i
0.	—2.120	1.128	0.0028	—0.472	0.003	0
0.1	—2.122	1.129	0.0025	—0.629	0.004	0.074
0.2	—2.124	1.131	0.0020	—0.768	0.005	0.157
0.3	—2.126	1.132	0.0018			0.254
0.4	—2.128	1.134	0.0015			0.335

Filling in the table:

Direct procedure. Using Table 93, compute m_i, k_i, h^2f_i by formulas (2). Taking into account that in the given problem $\alpha_0=1$, $\alpha_1=0$, $A=0$, we get

$$c_0=\frac{1}{m_0}=-0.472, \quad d_0=f_0h^2=0.003.$$

Then we successively find c_1, c_2, d_1, d_2.

Reverse procedure. From the boundary conditions we have $y_0=0$, $y_4=0.335$. The remaining values of y_i $(i=3, 2, 1)$ we find successively by formulas (3), for instance:

$$y_3=c_2(d_2-y_4)=0.768\,(0.005-0.335)=0.254.$$

Problems

1. Using the "passage" method, find to within 10^{-2} the solutions of the following differential equations, satisfying the boundary

conditions $y(0)=1$, $y(1)=e^{-1}+1=1.367$.

(a) $y''+\frac{x}{2}y'+(1+2\pi^2x^2)y=4x$, (b) $y''+(x-1)y'-3.125y=4x$,

(c) $y''+2xy'+2y=\frac{2(5-2x)}{(2-x)^3}$.

2. Find the approximate solutions of the following boundary value problem by the "passage" method

$$y''+f_j(x)y'+\cos(ax)y=2x^2+2x-4,\quad y(0)=0,\quad y(1)=0,$$
$$a=0.7+0.05\cdot k,\quad k=0,1,2,3,4$$

if the functions $f_j(x)$ $(j=1,2,3,4,5)$ are given by Table 95.

Table 95

Values of Functions $f_j(x)$

x_i	$f_1(x)$	$f_2(x)$	$f_3(x)$	$f_4(x)$	$f_5(x)$
0.0	−1.7930	−1.7012	−1.6184	−1.5432	−1.4747
0.2	−1.7863	−1.6877	−1.5994	−1.5200	−1.4480
0.4	−1.7832	−1.6776	−1.5838	−1.5000	−1.4246
0.6	−1.7838	−1.6709	−1.5714	−1.4832	−1.4043
0.8	−1.7878	−1.6673	−1.5630	−1.4692	−1.3869
1.0	−1.7953	−1.6668	−1.5555	−1.4581	−1.3722

3. Using the "passage" method find the solutions of the following equations on the interval [0, 1] with spacing $h=0.1$ for the boundary conditions $y(0)=y(1)=0$.

(a) $y''+(a+x^3)y'+(1-x^2)y=e^{1-bx^2}$,

(b) $y''+x^2y'+(a-x)y=\frac{x}{x^2+b}$,

(c) $y''+y'\sin ax+y=\frac{1}{b+\sin^2 ax}$, (d) $y''+\frac{y'}{\sqrt{x^2+b}}+ay=x$.

The parameters a and b take on the following values

$a=1+0.4\cdot k$, $k=0,1,2,3$, $b=2.5+0.5\cdot n$, $n=0,1,2,3,4,5$.

9.4 THE METHOD OF FINITE DIFFERENCES FOR SECOND-ORDER NONLINEAR DIFFERENTIAL EQUATIONS

Consider a nonlinear differential equation

$$y''=f(x,y,y') \tag{1}$$

for linear boundary conditions

$$\alpha_0y(a)-\alpha_1y'(a)=A,\quad \beta_0y(b)+\beta_1y'(b)=B. \tag{2}$$

Let us take on the interval $[a, b]$ a system of equally spaced points $x_0 = a$, $x_k = x_0 + kh$ $(k = 1, 2, \ldots, n-1)$ with a certain spacing $h = \frac{b-a}{n}$ and replace approximately the equation (1) and boundary conditions (2) by the system

$$\left.\begin{aligned} &\frac{y_{k+1} - 2y_k + y_{k-1}}{h^2} = f\left(x_k, y_k, \frac{y_{k+1} - y_{k-1}}{2h}\right) \quad (k = 1, 2, \ldots, n-1), \\ &\alpha_0 y_0 - \alpha_1 \frac{y_1 - y_0}{h} = A, \quad \beta_0 y_n + \beta_1 \frac{y_n - y_{n-1}}{h} = B. \end{aligned}\right\} \tag{3}$$

We obtain a nonlinear system of $n+1$ equations in $n+1$ unknowns y_k $(k = 0, 1, \ldots, n)$. Denote

$$\left.\begin{aligned} \Gamma_0(y) &= \alpha_0 y_0 - \alpha_1 \frac{y_1 - y_0}{h}, \\ \Gamma_n(y) &= \beta_0 y_n + \beta_1 \frac{y_n - y_{n-1}}{h}. \end{aligned}\right\} \tag{4}$$

The solution of system (3) is found by the method of iteration using the following formulas:

$$\left.\begin{aligned} &\frac{y_{k-1}^{(r+1)} - 2y_k^{(r+1)} + y_{k-1}^{(r+1)}}{h^2} = f\left(x_k, y_k^{(r)}, \frac{y_{k+1}^{(r)} - y_{k-1}^{(r)}}{2h}\right) \\ &\qquad\qquad (k = 1, 2, \ldots, n-1), \\ &\Gamma_0[y^{(r+1)}] = A, \quad \Gamma_n[y^{(r+1)}] = B. \end{aligned}\right\} \tag{5}$$

Here the superscript r indicates the number of approximation. We have to solve a system of linear algebraic equations for each interval (spacing) of iterations. Taking advantage of the special form of this system, we can give its solution in the explicit form (see [2]):

$$y_k^{(r+1)} = \frac{h}{\Delta}[A\beta_0(b-a) + A\beta_1 + \alpha_1 B] + \frac{k}{\Delta}(\alpha_0 B - A\beta_0) + h^2 \sum_{i=1}^{n-1} g_{ik} f_i^{(r)}, \tag{6}$$

where the numbers a, b, A, B, α_0, α_1, β_0, β_1 are known, and Δ and g_{ik} are computed by the formulas

$$\Delta = \frac{1}{h}[\alpha_0\beta_0(b-a) + \alpha_0\beta_1 + \alpha_1\beta_0],$$

$$g_{ik} = \begin{cases} \frac{1}{\Delta}\left(i\alpha_0 + \frac{\alpha_1}{h}\right)\left(k\beta_0 - \beta_0 n - \frac{\beta_1}{h}\right) & (i \leqslant k), \\ \frac{1}{\Delta}\left(k\alpha_0 + \frac{\alpha_1}{h}\right)\left(i\beta_0 - \beta_0 n - \frac{\beta_1}{h}\right) & (i > k). \end{cases} \tag{8}$$

Note that in the right-hand member of formula (6) only $f_i^{(r)}$ depends on the number of iteration.

Thus, the finding of the solution of system (3) is reduced to a rather simple iterative scheme.

The conditions of convergence of the indicated method are considered in [2].

Example 1. Using the method of finite differences, find (to within $0.5 \cdot 10^{-3}$) the solution of the boundary value problem

$$\left.\begin{aligned} y'' &= 2 + y^2, \\ y(0) &= y(1) = 0. \end{aligned}\right\} \tag{9}$$

Solution. Let us take $n = 10$, $h = 0.1$. For the given problem we have

$$a = 0, \quad b = 1, \quad \alpha_0 = 1, \quad \alpha_1 = 0, \quad \beta_0 = 1, \quad \beta_1 = 0, \quad A = B = 0,$$
$$f(x, y) = 2 + y^2.$$

For these values from formulas (6) to (8) we get

$$y_k^{(r+1)} = h^2 \sum_{i=1}^{n=1} g_{ik} f_i^{(r)}, \tag{10}$$

$$g_{ik} = \begin{cases} 0.1 \cdot i\,(k-n), & i \leqslant k, \\ 0.1 \cdot k\,(i-n), & i > k. \end{cases} \tag{11}$$

1. *Computing the coefficients* g_{ik}. From formula (11) we get for $i = 1$

$$g_{ik} = 0.1\,(k - 10).$$

Thus,

$$g_{11} = -0.9; \quad g_{12} = -0.8; \quad \ldots; \quad g_{19} = -0.1.$$

Then, from formula (11) we have for $i = 2$

$$g_{2k} = \begin{cases} 0.2\,(k-10), & 2 \leqslant k, \\ 0.1\,(-8), & k = 1. \end{cases}$$

Hence we find

$$g_{21} = -0.8; \quad g_{22} = -1.6; \quad g_{23} = -1.4; \quad \ldots; \quad g_{29} = -0.2.$$

For $i \geqslant 3$ computations are carried out analogously. All the results are entered in a separate table (Table 96).

2. *Choosing the initial approximation.* Let us choose the function $y^{(0)} = x(x-1)$, which is the solution of the equation $y'' = 2$, satisfying the boundary conditions $y(0) = y(1) = 0$. Enter in Table 97 the values of $x_k = 0.1 \cdot k$, the corresponding values of $y_k^{(0)} = x_k(x_k - 1)$ ($k = 1, 2, \ldots, 9$) and compute $f_k^{(0)} = 2 + [y_k^{(0)}]^2$.

3. *Computing the first approximation.* The values $y_k^{(1)}$ are found by formula (10) for $r = 0$:

$$y_k^{(1)} = 0.01 \sum_{i=1}^{9} g_{ik} f_i^{(0)}.$$

Table 96

Values of Coefficients g_{ik}

k \ i	1	2	3	4	5	6	7	8	9
1	−0.9	−0.8	−0.7	−0.6	−0.5	−0.4	−0.3	−0.2	−0.1
2	−0.8	−1.6	−1.4	−1.2	−1.0	−0.8	−0.6	−0.4	−0.2
3	−0.7	−1.4	−2.1	−1.8	−1.5	−1.2	−0.9	−0.6	−0.3
4	−0.6	−1.2	−1.8	−2.4	−2.0	−1.6	−1.2	−0.8	−0.4
5	−0.5	−1.0	−1.5	−2.0	−2.5	−2.0	−1.5	−1.0	−0.5
6	−0.4	−0.8	−1.2	−1.6	−2.0	−2.4	−1.8	−1.2	−0.6
7	−0.3	−0.6	−0.9	−1.2	−1.5	−1.8	−2.1	−1.4	−0.7
8	−0.2	−0.4	−0.6	−0.8	−1.0	−1.2	−1.4	−1.6	−0.8
9	−0.1	−0.2	−0.3	−0.4	−0.5	−0.6	−0.7	−0.8	−0.9

Table 97

Solving Problem (9)

r	k	1	2	3	4	5
	x_k	0.1	0.2	0.3	0.4	0.5
0	$x_k\,(x_k-1)$	−0.09	−0.16	−0.21	−0.29	−0.25
	$f_k^{(0)}$	2.0081	2.0256	2.0441	2.0576	2.0625
1	$y_k^{(1)}$	−0.0917	−0.1632	−0.2146	−0.2455	−0.2558
	$f_k^{(1)}$	2.0085	2.0266	2.0462	2.0602	2.0655
2	$y_k^{(2)}$	−0.0916	−0.1632	−0.2145	−0.2452	−0.2555

r	k	6	7	8	9
	x_k	0.6	0.7	0.8	0.9
0	$x_k\,(x_k-1)$	−0.24	−0.21	−0.16	−0.09
	$f_k^{(0)}$	2.0576	2.0441	2.0256	2.0081
1	$y_k^{(1)}$	−0.2455	−0.2146	−0.1632	−0.0917
	$f_k^{(1)}$	2.0602	2.0462	2.0266	2.0085
2	$y_k^{(2)}$	−0.2452	−0.2145	−0.1632	−0.0916

For $k=1$ we get

$$y_1^{(1)} = 0.01 \sum_{i=1}^{9} g_{i1} f_i^{(0)} = -0.01\,(0.9\cdot 2.0081 + 0.8\cdot 2.0256 + 0.7\cdot 2.0441 + 0.6\cdot 2.0576 + 0.5\cdot 2.0625 + 0.4\cdot 2.0576 + 0.3\cdot 2.0441 + 0.2\cdot 2.0256 + 0.1\cdot 2.0081) = -0.0917.$$

Computations for $k = 2, 3, \ldots, 9$ are carried out analogously. Write down the results in the row $y_k^{(1)}$ of Table 97, and find $f_k^{(1)} = 2 - (y_k^{(1)})^2$.

4. *Computing the second approximation.* The values $y_k^{(2)}$ are found by formula (10) for $r = 1$:

$$y_k^{(2)} = 0.01 \sum_{i=1}^{9} g_{ik} f_i^{(1)}.$$

In particular, for $k = 1$ we get

$$y_1^{(2)} = 0.01 \sum_{i=1}^{9} g_{i2} f_i^{(2)} = -0.01\,(0.8 \cdot 2.0085 + 1.6 \cdot 2.0266 + \\ + 1.4 \cdot 2.0462 + 1.2 \cdot 2.0602 + 1.0 \cdot 2.0655 + 0.8 \cdot 2.0602 + \\ + 0.6 \cdot 2.0462 + 0.4 \cdot 2.0266 + 0.2 \cdot 2.0085) = -0.0916.$$

Analogously, find $y_k^{(2)}$ for $k = 2, 3, \ldots, 9$, and enter the results in the row $y_k^{(2)}$.

Comparing the values of $y_k^{(1)}$ and $y_k^{(2)}$, we note that they differ only in the last digit. Hence, we may write approximately $y_k \approx y_k^{(2)}$.

9.5 GALERKIN'S METHOD

The above considered method of finite differences enables us to approximate the solution of a boundary value problem in the tabular form. Let us now treat two analytical methods which make it possible to find the approximate solution of a linear boundary value problem in the form of an analytic expression, namely, Galerkin's method and the collocation method.

Suppose we have a linear boundary value problem (3), (4) of Sec. 9.1.

Let us introduce the following notation

$$\left.\begin{aligned} L[y] &= y'' + p(x)\,y' + q(x)\,y, \\ \Gamma_a[y] &= \alpha_0 y(a) + \alpha_1 y'(a), \\ \Gamma_b[y] &= \beta_0 y(b) + \beta_1 y'(b). \end{aligned}\right\} \tag{1}$$

Let on an interval $[a, b]$ there be given a system of *basis functions*

$$u_0(x), \quad u_1(x), \quad \ldots, \quad u_n(x), \quad \ldots, \tag{2}$$

satisfying the following conditions:

(1) System (2) is orthogonal, i.e.

$$\left.\begin{aligned} &\int_a^b u_i(x)\,u_j(x)\,dx = 0 \quad \text{for } i \neq j, \\ &\int_a^b u_i^2(x)\,dx \neq 0. \end{aligned}\right\} \tag{3}$$

(2) System (2) is a complete one, i.e. there is no other non-zero function which is orthogonal to all the functions $u_i(x)$ ($i = 0, 1, 2, \ldots$).

(3) The finite system of basis functions $\{u_i(x)\}$ ($i = 0, 1, \ldots, n$) is chosen so that the function $u_0(x)$ satisfies the non-homogeneous boundary conditions

$$\Gamma_a[u_0] = A, \quad \Gamma_b[u_0] = B, \tag{4}$$

and the functions $u_i(x)$ ($i = 1, 2, \ldots, n$) satisfy the homogeneous boundary conditions

$$\Gamma_a[u_i] = \Gamma_b[u_i] = 0 \quad (i = 1, 2, \ldots, n). \tag{5}$$

We shall look for the solution of the boundary problem (3), (4) of Sec. 9.1 in the form

$$y(x) = u_0(x) + \sum_{i=1}^{n} c_i u_i(x). \tag{6}$$

It follows from conditions (4), (5) that this function satisfies the boundary conditions (4) of Sec. 9.1.

Consider the expression called the *residual*:

$$R(x, c_1, c_2, \ldots, c_n) = L[u_0] + \sum_{i=1}^{n} c_i L[u_i] - f(x). \tag{7}$$

Choose the coefficients c_i so that to obtain the least value of the integral of the squared residual.

$$\int_a^b R^2(x, c_1, c_2, \ldots, c_n)\, dx. \tag{8}$$

It is proved (see [2], [13], [18]) that this is achieved only if the residual $R(x, c_1, c_2, \ldots, c_n)$ is orthogonal to all the basis functions u_i.

Let us write down the condition of orthogonality:

$$\int_a^b u_k(x)\, R(x, c_1, c_2, \ldots, c_n)\, dx = 0 \quad (k = 1, 2, \ldots, n)$$

or in full

$$\sum_{i=1}^{n} c_i \int_a^b u_k(x)\, L[u_i]\, dx = \int_a^b u_k(x) \{f(x) - L[u_0]\}\, dx. \tag{9}$$

Thus, we obtained a system of linear algebraic equations with respect to c_i.

Note that in choosing the basis functions the condition of orthogonality (3) is not obligatory if the coefficients are chosen proceeding from the minimality condition of the integral (8). For instance, taking for the basis a complete system of functions orthogonal

on the interval $[a, b]$, we may choose as the basis functions the linear combinations of functions from this system. It is only necessary and sufficient that the chosen functions be linearly independent on the interval $[a, b]$.

Example 1. Using Galerkin's method, approximate the solution of the equation

$$y'' - y' \cos x + y \sin x = \sin x, \tag{10}$$

satisfying the boundary conditions

$$y(-\pi) = y(\pi) = 2. \tag{11}$$

Solution. As the system of the basis functions $u_i(x)$ $(i = 0, 1, 2, 3, 4)$ choose the following trigonometric functions:

$$u_0 = 2, \quad u_1 = \sin x, \quad u_2 = \cos x + 1, \quad u_3 = \sin 2x, \quad u_4 = \cos 2x - 1. \tag{12}$$

They are linearly independent on the interval $[-\pi, \pi]$, the function u_0 satisfying the boundary condition (11), and the rest of them, the zero boundary conditions. Let us look for a solution in the form

$$y = u_0(x) + \sum_{i=1}^{4} c_i u_i(x).$$

We find $L[u_i] = (i = 0, 1, 2, 3, 4)$:

$$L[u_0] = 2 \sin x,$$

$$L[u_1] = -\sin x - \cos 2x,$$

$$L[u_2] = \sin x - \cos x + \sin 2x,$$

$$L[u_3] = -\frac{1}{2} \cos x - 4 \sin 2x - \frac{3}{2} \cos 3x,$$

$$L[u_4] = -\frac{1}{2} \sin x - 4 \cos 2x + \frac{3}{2} \cos 3x,$$

$$f(x) - L[u_0] = -\sin x.$$

Compute the coefficients of system (9), using the following notation:

$$a_{ik} = \int_a^0 u_k(x) L[u_i] \, dx, \quad b_k = \int_a^b u_k(x) \{f(x) - L[u_0]\} \, dx,$$

and taking into consideration orthogonality of the system of trigonometric functions $(1, \sin x, \cos x, \sin 2x, \cos 2x, \ldots)$

$$b_1 = -\int_{-\pi}^{\pi} \sin^2 x \, dx = -\pi, \quad b_2 = b_3 = b_4 = 0,$$

$$a_{11}=-\int_{-\pi}^{\pi}\sin^2 x\,dx=-\pi,\quad a_{12}=\pi,\quad a_{13}=0,\quad a_{14}=-\frac{\pi}{2},$$

$$a_{21}=0,\quad a_{22}=-\int_{-\pi}^{\pi}\cos^2 x\,dx=-\pi,\quad a_{23}=-\frac{\pi}{2},\quad a_{24}=0,$$

$$a_{31}=0,\quad a_{32}=\int_{-\pi}^{\pi}\sin^2 2x\,dx=\pi,\quad a_{33}=-4\pi,\quad a_{34}=0,$$

$$a_{41}=-\int_{-\pi}^{\pi}\cos^2 2x\,dx=-\pi,\quad a_{42}=0,\quad a_{43}=0,\quad a_{44}=-4\pi.$$

On reducing we arrive at the following system

$$\begin{aligned} c_1-c_2 \qquad\quad &+0.5c_4=1,\\ c_2+0.5c_3 \qquad\quad &=0,\\ c_2-4c_3 \qquad\quad &=0,\\ c_1 \qquad\quad &+4c_4=0, \end{aligned}$$

whence we get $c_1=\frac{8}{7}$, $c_2=c_3=0$, $c_4=-\frac{2}{7}$. Thus we have

$$y\approx 2+\frac{8}{7}\sin x+\frac{4}{7}\sin^2 x.$$

For the comparison purposes Table 98 gives the values of the obtained approximate solution and the exact solution of $y=e^{\sin x}+1$.

Table 98

Approximate and Exact Solutions of Problem (10), (11)

x_i	$-\frac{\pi}{2}$	0	$\frac{\pi}{2}$
y_i	1.429	2	3.714
y	1.368	2	3.718

Example 2. Using Galerkin's method, approximate the solution of the equation

$$y''+y+x=0, \tag{13}$$

satisfying the boundary conditions

$$y(0)=y(1)=0. \tag{14}$$

Solution. Choose the following functions as the system of basis functions u_i:

$$u_0(x) \equiv 0, \quad u_1(x) = x(1-x), \quad u_2(x) = x^2(1-x).$$

They are linearly independent and satisfy zero boundary conditions. We look for an approximate solution of the problem in the form

$$y = c_1 x(1-x) + c_2 x^2(1-x) = x(1-x)(c_1 + c_2 x).$$

Substituting y in the left-hand member of equation (13), we get the residual

$$R(x, c_1, c_2) = -2c_1 + c_2(2-6x) + x(1-x)(c_1 + c_2 x) + x.$$

Taking into account orthogonality of the function R with respect to the functions $u_1(x)$ and $u_2(x)$, we get the following system

$$\int_0^1 (x-x^2) R(x, c_1, c_2)\, dx = 0,$$

$$\int_0^1 (x^2-x^3) R(x, c_1, c_2)\, dx = 0.$$

Substituting the value of R in this system and computing the integrals, we get a system of linear algebraic equations for determining the coefficients c_1 and c_2:

$$\frac{3}{10} c_1 + \frac{3}{20} c_2 = \frac{1}{12},$$

$$\frac{3}{20} c_1 + \frac{13}{105} c_2 = \frac{1}{20}.$$

Solving this system, we get

$$c_1 = \frac{71}{369}, \quad c_2 = \frac{7}{41}.$$

Thus,

$$y(x) = x(1-x)\left(\frac{71}{369} + \frac{7}{41} x\right).$$

For the comparison purposes Table 99 gives the values of the obtained approximate solution and the exact solution of

$$y = \frac{\sin x}{\sin 1} - x.$$

Note. The above examples show that the adequate choice of the basis functions makes it possible to approximate the solution of a boundary value problem in an analytic form.

If the functions $p(x)$, $q(x)$, $f(x)$ in equation (3) of Sec. 9.1 are rather complicated, then the computation of the coefficients of system (9) becomes cumbersome. In such cases it is advisable to use either the difference method, or the collocation method which is dealt with in the next section.

Table 99

Approximate and Exact Solutions Problem (13), (14)

x_i	0.25	0.50	0.75
y_i	0.044	0.069	0.060
y	0.044	0.070	0.060

Problems

Applying Galerkin's method, approximate the solutions of the following boundary value problems:

1. $y''-y'\cos x+y\sin x=f(x)$, $y(-\pi)=y(\pi)=2$.

(a) $f(x)=\cos x$, (b) $f(x)=\sin x$, (c) $f(x)=\cos 2x$.

2. $y''-2xy'+2y=f(x)$, $y(0)=0$, $y'(1)=1$.

(a) $f(x)=x$, (b) $f(x)=3x^2+x-1$, (c) $f(x)=5x^3-3x+x$,

(d) $f(x)=\frac{1}{2}\left(5x^3-3x^2-\frac{1}{2}x+1\right)$.

9.6 COLLOCATION METHOD

We look for the solution of boundary value problem (3), (4) of Sec. 9.1 in the form

$$y(x)=u_0(x)+\sum_{i=1}^{n}c_iu_i(x), \tag{1}$$

where $u_i(x)$ $(i=0, 1, \ldots, n)$ are linearly independent functions satisfying the conditions (4), (5) of Sec. 9.5. Let us require that the residual

$$R(x, c_1, c_2, \ldots, c_n)=L[y]-f(x)=L[u_0]-f(x)+\sum_{i=1}^{n}c_iL[u_i] \tag{2}$$

vanish for a certain system of points $x_1, x_2, \ldots, x_n$ of the interval $[a, b]$ called the *collocation points* (the number of such points must equal the number of the coefficients c_1 in the expression (1). Then for determining $c_1, c_2, \ldots, c_n$ we get the following system of equations

$$\left.\begin{aligned} R(x_1, c_1, c_2, \ldots, c_n)&=0,\\ R(x_2, c_1, c_2, \ldots, c_n)&=0,\\ \ldots\ldots\ldots\ldots&\\ R(x_n, c_1, c_2, \ldots, c_n)&=0. \end{aligned}\right\} \tag{3}$$

The collocation method may also be used for approximating the solution of nonlinear differential equations

$$y'' = f(x, y, y') \tag{4}$$

with linear boundary conditions (4) of Sec. 9.1. In this case the residual has the form

$$R(x) = y'' - f(x, y, y'), \tag{5}$$

and system (6.3) turns out to be a system of nonlinear algebraic equations with respect to the unknowns $c_1, c_2, \ldots, c_n$ (see Example 2).

Example 1. Using the collocation method, approximate the solution of the equation

$$y'' + (1 + x^2) y + 1 = 0 \tag{6}$$

with the boundary conditions

$$y(-1) = y(1) = 0. \tag{7}$$

Solution. Judging by the form of equation (6) and boundary value conditions (7), we can judge as to whether the functions are even or odd. Therefore, let us take the polynomials $u_0(x) = 0$, $u_1(x) = 1 - x^2$, $u_2(x) = x^2(1 - x^2)$. It is easy to note that the conditions (7) are fulfilled for them.

We shall seak for a solution of problem (6), (7) in the form

$$y = c_1(1 - x^2) + c_2 x^2 (1 - x^2),$$

taking for collocation points

$$x_0 = 0, \quad x_1 = \frac{1}{2}.$$

Form the residual $R(x)$:

$$R(x) = -2c_1 + c_2(2 - 12x^2) + (1 + x^2)[c_1(1 - x^2) + c_2(x^2 - x^4)] + \\ + 1 = 1 - c_1(1 + x^4) + c_2(2 - 11x^2 - x^6).$$

Substituting the values $x_0 = 0$, $x_1 = 1/2$, we get the system

$$1 - c_1 + 2c_2 = 0, \quad 1 - \frac{17}{16} c_1 - \frac{49}{64} c_2 = 0,$$

whence we find $c_1 = 0.957$, $c_2 = -0.022$. Hence, we have an approximate solution

$$y \approx 0.957(1 - x^2) - 0.022x^2(1 - x^2).$$

Example 2. Using the collocation method, approximate the solution of the equation

$$y'' = 2x + y^2 \tag{8}$$

with the boundary conditions

$$y(0) = y(1) = 0.$$

Solution. Take the following functions as the base ones

$$u_0 = 0, \quad u_1 = x(1-x), \quad u_2 = x^2(1-x)$$

and look for a solution in the form $y = c_1u_1(x) + c_2u_2(x)$. Form the residual $R(x)$:

$$R(x) = c_1u_1'' + c_2u_2'' - (c_1^2u_1^2 + 2c_1c_2u_1u_2 + c_2^2u_2^2) - 2x,$$

where $u_1'' = -2$, $u_2'' = 2 - 6x$.

Take the collocation points: $x_1 = 0.25$, $x_2 = 0.75$.

Computing $R(x)$ at the collocation points, we get the system of non-linear equations

$$\left.\begin{aligned} -2c_1 + 0.5c_2 &= 0.5 + (0.0352c_1^2 + 0.0176c_1c_2 + 0.0022c_2^2), \\ -2c_1 - 2.5c_2 &= 1.5 + (0.0352c_1^2 + 0.0528c_1c_2 + 0.0198c_2^2). \end{aligned}\right\} \quad (9)$$

Write this system as follows

$$\left.\begin{aligned} c_1 &= -\frac{1}{3} - \frac{5}{6}(0.0211c_1^2 + 0.0141c_1c_2 + 0.0031c_2^2), \\ c_2 &= -\frac{1}{3} - \frac{1}{3}(0.0352c_1c_2 + 0.0176c_2^2) \end{aligned}\right\} \quad (10)$$

and solve it by the iterative method using the formulas

$$c_{1,\,k+1} = -\frac{1}{3} - \frac{5}{6}(0.0211c_{1,\,k}^2 + 0.0141c_{1,\,k}c_{2,\,k} + 0.0031c_{2,\,k}^2),$$

$$c_{2,\,k+1} = -\frac{1}{3} - \frac{1}{3}(0.0352c_{1,\,k}c_{2,\,k} + 0.0176c_{2,\,k}^2).$$

Here the subscript k indicates the number of approximation.

First approximation:

$$c_{1,\,1} = -\frac{1}{3}, \quad c_{2,\,1} = -\frac{1}{3},$$

$$c_{1,\,1}^2 = c_{2,\,1}^2 = c_{1,\,1}c_{2,\,1} = \frac{1}{9}.$$

Second approximation:

$$c_{1,\,2} = -\frac{1}{3} - \frac{5}{6} \cdot \frac{1}{9}(0.0211 + 0.0141 + 0.0031) = -0.3369,$$

$$c_{2,\,2} = -\frac{1}{3} - \frac{1}{3} \cdot \frac{1}{9}(0.0352 + 0.0176) = -0.3353,$$

$$c_{1,\,2}^2 = 0.1135, \quad c_{2,\,2}^2 = 0.1124, \quad c_{1,\,2}c_{2,\,2} = 0.1130.$$

Third approximation:

$$c_{1,\,3} = -\frac{1}{3} - \frac{5}{6}(0.211 \cdot 0.1135 + 0.0141 \cdot 0.1130 + 0.0031 \cdot 0.1124) = -0.3369,$$

$$c_{2,\,3} = -\frac{1}{3} - \frac{1}{3}(0.0352 \cdot 0.1130 + 0.0176 \cdot 0.1124) = -0.3353.$$

Since the values c_1, c_2 in the second and third approximations coincide, we may write accurate to 10^{-4}

$$c_1 = -0.3369, \quad c_2 = -0.3353.$$

Thus, we have an approximate solution of equation (8)

$$y = -x(1-x)(0.3369 + 0.3353x).$$

Note. As is clear from the above considered examples, the collocation method leads to simpler computations, as compared with Galerkin's method.

Problems

Applying the collocation method, approximate the solutions of the following boundary value problems.

1. $y'' + x^2y' - xy = f(x)$, $y(0) = y(1) = 0$.

(a) $f(x) = e^x$, (b) $f(x) = e^{x^2}$, (c) $f(x) = \sin x$, (d) $f(x) = \cos x$, (e) $f(x) = \tan x$.

2. $y'' + y' - \frac{y}{x} = f(x)$, $y(0) = 0$, $y'(1) = 1$.

(a) $f(x) = x^2 - \frac{3}{4}x + \frac{1}{8}$, (b) $f(x) = 8x^2 - 8x + \frac{3}{2}$,

(c) $f(x) = 4x^2 - x + \frac{3}{2}$.

10

NUMERICAL SOLUTION OF EQUATIONS WITH PARTIAL DERIVATIVES AND OF INTEGRAL EQUATIONS

10.1 THE NET METHOD

The *net method*, or the *method of finite differences*, is one of the most widely spread at present methods of numerical solution of equations with partial derivatives. It is based on the idea of replacing the derivatives by finite-difference relations. We shall confine ourselves to the case of two independent variables. Let a certain domain G with the boundary Γ lie in the plane xOy (Fig. 4).

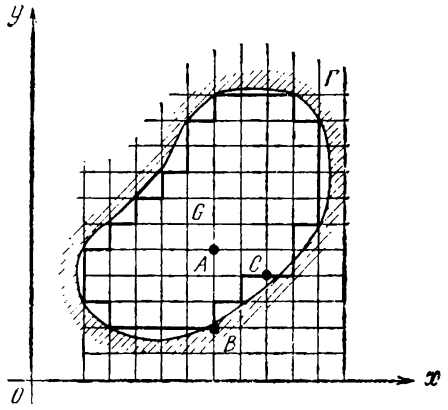

Fig. 4

Let us construct in this plane two families of parallel straight lines:

$$x = x_0 + ih$$
$$(i = 0, \pm 1, \pm 2, \ldots),$$
$$y = y_0 + kl$$
$$(k = 0, \pm 1, \pm 2, \ldots).$$

The points of intersection of these straight lines are called *mesh points*. Two mesh points are termed *neighbouring* ones if the distance between them in the direction of the axis Ox or Oy is equal to the net spacing h or l, respectively. Let us single out the mesh points belonging to the domain $G+\Gamma$, and also some points not belonging to this domain but located at a distance less than the spacing from the boundary Γ. The points whose all four neighbouring points belong to the isolated set of mesh points are referred to as *interior* ones (say, point A in Fig 4). The rest of the singled out points are called *boundary* ones (as points B, C in Fig. 4).

Let us denote the values of the required function $u=u(x, y)$ at the mesh points by $u_{ik}=u(x_0+ih, y_0+kl)$. At each interior point (x_0+ih, y_0+kl) replace the partial derivatives by difference relations:

$$\left(\frac{\partial u}{\partial x}\right)_{ik}\approx\frac{u_{i+1,k}-u_{i-1,k}}{2h}, \quad \left(\frac{\partial u}{\partial y}\right)_{ik}\approx\frac{u_{i,k+1}-u_{i,k-1}}{2l}, \tag{1}$$

for the boundary points we are forced to use less accurate formulas of the form

$$\left.\begin{aligned}\left(\frac{\partial u}{\partial x}\right)_{ik}&\approx\frac{u_{i+1,k}-u_{ik}}{h},\\ \left(\frac{\partial u}{\partial y}\right)_{ik}&\approx\frac{u_{i,k+1}-u_{ik}}{l}.\end{aligned}\right\} \tag{2}$$

Partial derivatives of the second order are replaced analogously, for instance:

$$\left.\begin{aligned}\left(\frac{\partial^2 u}{\partial x^2}\right)_{ik}&\approx\frac{u_{i+1,k}-2u_{ik}+u_{i-1,k}}{h^2},\\ \left(\frac{\partial^2 u}{\partial y^2}\right)_{ik}&\approx\frac{u_{i,k+1}-2u_{ik}+u_{i,k-1}}{l^2}.\end{aligned}\right\} \tag{3}$$

The indicated substitutions of derivatives at each mesh point enable us to reduce the solution of equations with partial derivatives to that of a system of difference equations.

10.2 THE NET METHOD FOR DIRICHLET'S PROBLEM

The first boundary value problem, or Dirichlet's problem for Poisson's equation

$$\Delta u=\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{dy^2}=f(x, y) \tag{1}$$

is stated as follows: find the function $u=u(x, y)$ satisfying inside a certain domain G the equation (1), and at the boundary Γ, the

condition

$$u|_\Gamma = \varphi(x, y), \tag{2}$$

where $\varphi(x, y)$ is the given continuous function.

Choosing h and l along x and y, respectively, construct a net

$$x_i = x_0 + ih \quad (i = 0, \pm 1, \pm 2, \ldots),$$
$$y_k = y_0 + kl \quad (k = 0, \pm 1, \pm 2, \ldots)$$

and at each interior mesh point (x_i, y_k) replace the derivatives $\frac{\partial^2 u}{\partial x^2}$, $\frac{\partial^2 u}{\partial y^2}$ by finite-difference relations (3) of Sec. 10.1 and equation (1) by the finite-difference equations

$$\frac{u_{i+1,k} - 2u_{ik} + u_{i-1,k}}{h^2} + \frac{u_{i,k+1} - 2u_{ik} + u_{i,k-1}}{l^2} = f_{ik}, \tag{3}$$

where $f_{ik} = f_k(x_i, y_k)$.

Equations (3) together with the values of u_{ik} at boundary points form a system of linear algebraic equations with respect to the values of the function $u(x, y)$ at points (x_i, y_k). For a rectangular domain and for $l = h$ this system has the simplest form. In this case equations (2.3) are written as follows:

$$u_{i+1,k} + u_{i-1,k} + u_{i,k+1} + u_{i,k-1} - 4u_{ik} = h^2 f_{ik}, \tag{4}$$

and the values at boundary points are exactly equal to those of the boundary function. For $f(x, y) = 0$ equation (1) is called the *Laplace equation*, and the corresponding finite-difference equations have the form

$$u_{ik} = \frac{1}{4}(u_{i+1,k} + u_{i-1,k} + u_{i,k+1} + u_{i,k-1}). \tag{5}$$

In setting up equations (4) and (5) use was made of the scheme of mesh points shown in Fig. 5. Here and elsewhere indicated in figures are only the subscripts of the mesh points. For instance, the point (x_i, y_k) is designated as (i, k). Sometimes it turns out to be more convenient to use the scheme given in Fig. 6. In this case to Laplace's equation correspond the following finite-difference equations:

$$u_{ik} = \frac{1}{4}(u_{i-1,k-1} + u_{i+1,k-1} + u_{i-1,k+1} + u_{i+1,k+1}), \tag{6}$$

and for Poisson's equation we have

$$u_{ik} = \frac{1}{4}(u_{i-1,k-1} + u_{i+1,k-1} + u_{i-1,k+1} + u_{i+1,k+1}) + \frac{h^2}{2} f_{ik}.$$

The error due to the replacement of the differential equation by a difference equation, i. e. the remainder term R_{ik} for the Laplace

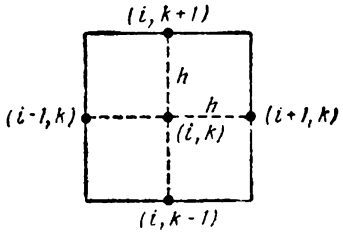

Fig. 5

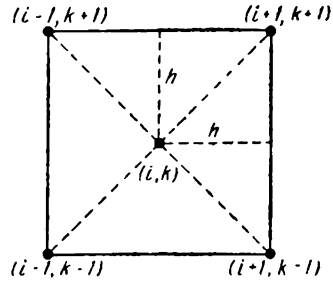

Fig. 6

equation is estimated by the inequality

$$|R_{ik}| \leqslant \frac{h^2}{6} M_4,$$

where

$$M_4 = \max_G \left\{ \left| \frac{\partial^4 u}{\partial x^4} \right|, \quad \left| \frac{\partial^4 u}{\partial y^4} \right| \right\}.$$

The error of the approximate solution obtained by the difference method is made up of the following three errors:

(1) the error due to the replacement of the differential equation by a difference equation;

(2) the error of approximating the boundary conditions;

(3) the error resulting from the fact that a system of difference equations is solved by the approximate method.

Example 1. Consider the problem of stationary distribution of heat in an isolated flat square plate whose side is equal to 1 if its boundary is kept at a constant temperature.

As is known (see [52]), the function $u(x, y)$, governing temperature distribution, is the solution of Laplace's equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

for appropriate boundary conditions. For the problem under consideration the boundary conditions are given in Fig. 7.

Solution. Construct a net with $h = \frac{1}{4}$, thus obtaining nine interior points (Fig. 7). Write finite-difference equations for these points.

By virtue of symmetry of the boundary conditions we have

$$u_{11} = u_{31}, \quad u_{12} = u_{32}, \quad u_{13} = u_{33}. \tag{7}$$

This reduces the number of unknown values of the function u at the interior points to six. Thus, there is no need to write finite-

difference equations for the mesh points (3, 1), (3, 2), (3, 3). For the remaining six interior points (1, 1), (2, 1), (1, 2), (2, 2), (1, 3), (2, 3) we get, respectively, six equations:

$$\left.\begin{array}{ll} u_{01}+u_{21}+u_{10}+u_{12}-4u_{11}=0, & u_{11}+u_{31}+u_{20}+u_{22}-4u_{21}=0, \\ u_{02}+u_{22}+u_{11}+u_{13}-4u_{12}=0, & u_{12}+u_{32}+u_{21}+u_{23}-4u_{22}=0, \\ u_{03}+u_{23}+u_{12}+u_{14}-4u_{13}=0, & u_{13}+u_{33}+u_{22}+u_{24}-4u_{23}=0. \end{array}\right\} \quad (8)$$

The twelve values of the function at the boundary points are also covered by these equations. We take these values from the boundary conditions

$$\left.\begin{array}{l} u_{i0}=0 \quad (i=1, 2, 3), \quad u_{0j}=0 \quad (j=1, 2, 3), \\ u_{14}=u_{24}=u_{32}=10\,000. \end{array}\right\} \quad (9)$$

Not that at the rest of the mesh points the boundary conditions are not used.

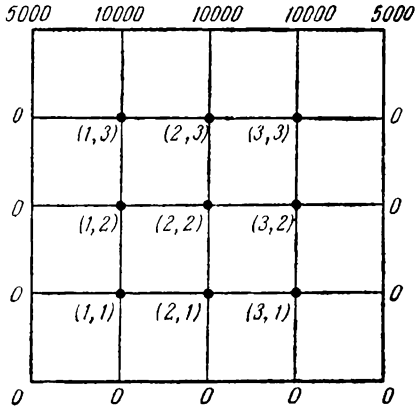

Fig. 7

Finally, taking into account the conditions (7), (9), we get the system

$$\begin{array}{ll} u_{21}+u_{12}-4u_{11}=0, & 2u_{11}+u_{22}-4u_{21}=0, \\ u_{22}+u_{11}+u_{13}-4u_{12}=0, & 2u_{12}+u_{21}+u_{23}-4u_{22}=0, \\ u_{23}+u_{12}-4u_{13}=-10\,00. & 2u_{13}+u_{22}-4u_{23}=-10\,000. \end{array}$$

Solving this system by the Gaussian method, we obtain

$$\begin{array}{llll} u_{11}=714, & u_{21}=982, & u_{12}=1875, & u_{22}=2500, \\ & u_{13}=4286, & u_{23}=5268. & \end{array}$$

Example 2. As is known (see [42]), the problem of elastic deformation of a square plate under the action of a constant force is

reduced to solving Poisson's equation

$$\Delta u = -1$$

with zero boundary values. Find the solution of this problem by the net method, taking the side of the square equal to 1 and the spacing h to $\frac{1}{4}$.

Solution. Note that in this case we have a complete symmetry of the values of the sought-for function, since all the boundary conditions are zero ones, and the function $f(x, y)$ is constant. Therefore, it is sufficient to set up finite-difference equations for a quarter of the square, i.e. for the points (1, 1), (2, 1), (1, 2), and (2, 2) (see Fig. 7). Taking into account the zero boundary conditions, we get the following equations:

$$u_{21}+u_{12}-4u_{11}=-0.0625, \quad 2u_{11}+u_{22}-4u_{21}=-0.0625,$$
$$u_{22}+2u_{11}-4u_{12}=-0.0625, \quad 2u_{12}+2u_{21}-4u_{22}=-0.0625.$$

By virtue of symmetry of the solution ($u_{12}=u_{21}$) the obtained system is reduced to the system of three equations

$$\begin{aligned} -4u_{11}+2u_{12} &= -0.0625, \\ 2u_{11}-4u_{12}+u_{22} &= -0.0625, \\ 4u_{12}-4u_{22} &= -0.0625. \end{aligned}$$

Solving this system, we get

$$u_{11}=0.0429, \quad u_{12}=u_{21}=0.0547, \quad u_{22}=0.0703.$$

Problems

1. Applying the net method, find the solution of Laplace's equation at the points p, q, r, s of the square with the boundary conditions indicated in Fig. 8 for $\alpha=0.9+0.1\cdot k$, $k=0, 1, 2$, $\beta=1.01+0.01\cdot n$, $n=0, 1, 2, 3, 4$.

2. Applying the net method with the spacing $h=\frac{1}{4}$, find the solution of Laplace's equation in a square with the vertices $A(0, 0)$, $B(0, 1)$, $C(1, 1)$, $D(1, 0)$. The boundary conditions are given in Table 100.

10.3 ITERATIVE METHOD FOR SOLVING A SYSTEM OF FINITE-DIFFERENCE EQUATIONS

A direct solution of a system of finite-difference equations by the method of successive elimination for a large number of mesh points turns out to be too cumbersome. Here it is advisable to

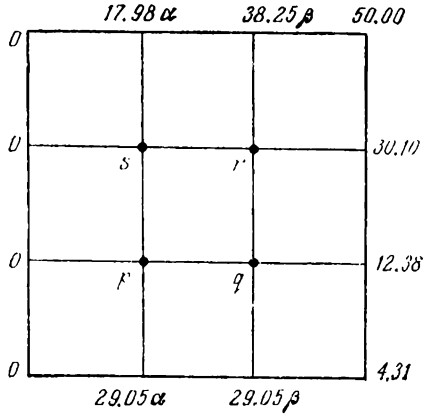

Fig. 8

Table 100

Boundary Conditions

Variants	$u\|_{AB}$	$u\|_{BC}$	$u\|_{CD}$	$u\|_{AD}$
(a)	$30y$	$30(1-x^2)$	0	0
(b)	$30y$	$30\cos\frac{\pi x}{2}$	$30\cos\frac{\pi y}{2}$	0
(c)	$50y(1-y^2)$	0	0	$50\sin\pi x$
(d)	$20y$	20	$20y^2$	$50x(1-x)$
(e)	0	$50x(1-x)$	$50y(1-y^2)$	$50x(1-x)$
(f)	$30\sin\pi y$	$20x$	$20y$	$30x(1-x)$
(g)	$30(1-y)$	$20\sqrt{x}$	$20y$	$30(1-x)$
(h)	$50\sin\pi y$	$30\sqrt{x}$	$30y^2$	$50\sin\pi x$
(i)	$40y^2$	40	40	$40\sin\frac{\pi x}{2}$
(j)	50	$50(1-x)$	0	$60x$, $0 \leqslant x < 1/2$ $60(1-x)$, $1/2 \leqslant x \leqslant 1$

use more convenient iterative methods of solution which take into account the special form of such systems and turn out to be suitable for realizing on a computer (see [5], [60]).

Consider one of the simplest methods, namely, Liebmann's process of averaging for systems (5) of Sec. 10.2 (see [13], [35], [39]) *.

* For other effective methods and complete bibliography see [43].

According to the Liebmann method, the computations are carried out in the following way: choosing the initial approximations $u_{ij}^{(0)}$, we determine the successive approximations for the interior points of a net domain by the formula

$$u_{ij}^{(k+1)} = \frac{1}{4}\left[u_{i+1,\,j}^{(k)} + u_{i-1,\,j}^{(k)} + u_{i,\,j+1}^{(k)} + u_{i,\,j-1}^{(k)}\right], \quad (k = 0, 1, 2, \ldots). \quad (1)$$

To obtain the initial approximations two methods can be applied:

(1) get the values of $u_{ij}^{(0)}$ at the interior mesh points by interpolation using the known boundary values (see Example 1);

2) form a system of finite-difference equations for a net of a larger spacing and solve it by the method of eliminating, and then interpolate the obtained values with respect to the given net (see Example 2).

It is proved (see [18], [37], [52]) that for any spacing h the Liebmann process converges to the exact solution independent of the choice of the initial values, i.e. there exists the limit

$$\lim_{k \to \infty} u_{ij}^{(k)} = u_{ij}.$$

The iterative process will converge much more rapidly if in computing the subsequent arithmetic means both the values of the previous approximation, and newly found values are used (Seidel's method, Example 2). Iterations are usually continued till the required number of decimal digits coincide in two successive approximations. The error of approximating the solution of the Laplace equation can be estimated by the Runge principle (see [2], [35]), according to which the error ε_h of the approximate solution u_h obtained for the spacing h, is given by the approximate formula

$$\varepsilon_h \approx \frac{u_h - u_{2h}}{3}, \quad (2)$$

where u_{2h} is the approximate solution obtained for the spacing $2h$. Note that the mentioned method of iterations calls for a standard operation of averaging carried out at each interior point, therefore it turns out to be very suitable for programming on a computer. When calculating on key computers it is useful to prepare a sufficient number of *computational patterns* (see [13], [35]).

Example 1. Find the solution of the Laplace equation for a square with the boundary conditions indicated in Fig. 9.

Solution. Figure 10 shows the computational pattern for the given problem. It is constructed in the following way. Each mesh point is replaced by a square. Written in the squares corresponding to the boundary points are the given boundary values. Since in the process of iterations the boundary values remain unchanged, the rest of the patterns are given in the form of a square 5×5 (Table 101) which are attached to the basic pattern (Fig. 10).

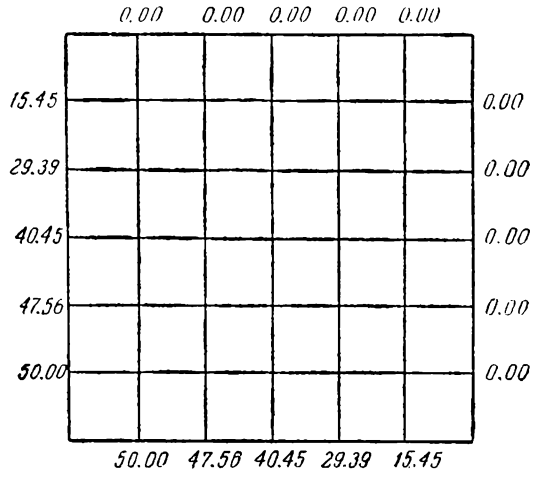

Fig. 9

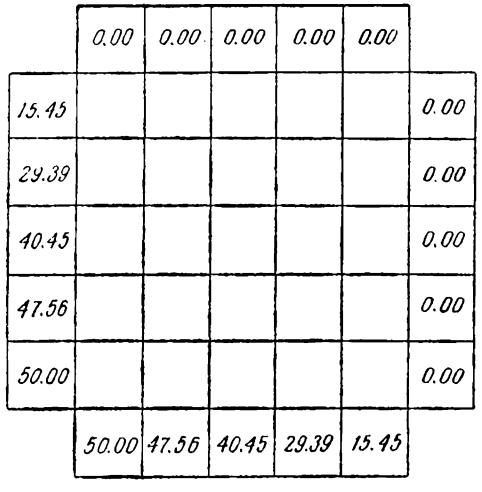

Fig. 10

Filling in the patterns:

1. *Computing the initial approximation.* Interpolate the boundary values to the interior points in the following way. Begin with the upper row. Let us consider that the function $u(x, y)$ decreases linearly from 15.45 to 0. This means that for the initial value

Table 101

Values of the Successive Approximations (Example 1)

Pattern 1

12.88	10.30	7.72	5.15	2.58
24.54	19.69	14.85	10 00	5.15
34.05	27.65	21.25	14.85	7.72
40.92	34.29	27.65	19.69	10.30
45.46	40.92	34.05	24.54	12.88

Pattern 2

12.57	10.07	7.58	5.08	2.58
24.00	19.34	14.66	10.00	
33.39	27.32	21.25		
40.34	34.28			
45.46				

Pattern 3

12.38	9.87	7.45	5.04	2.54
23.67	19.01	14.54	9.87	
33.03	27.06	20.99		
40.17	33.83			
45.17				

Pattern 4

12 25	9.71	7.36	4.96	2.52
23.45	18.78	14.33	9.79	
32.81	26.72	29.80		
39.90	33.62			
45.08				

Pattern 5

12.15	9.60	7.25	4.92	2.48
23.32	18.55	14.18	9.64	
32.63	26.51	20.52		
39.78	33.31			
44.95				

Pattern 6

12.09	9.49	7 18	4.84	2 46
23.18	18.40	13.99	9.55	
32.52	26.25	20.34		
39 61	33.14			
44.89				

Pattern 7

12.03	9.42	7.08	4.80	2.42
23.10	18.23	13.87	9.42	
32.37	26.10	20.12		
39.53	32.93			
44.80				

Pattern 8

11.99	9.34	7.02	4.73	2.40
23 00	18.12	13.71	9.34	
32.30	25.91	19.98		
39.42	32.82			
44.76				

Pattern 9

11.95	9.28	6.94	4.69	2.36
22.95	17.99	13.62	9.22	
32.20	25.80	19.81		
39.36	32.66			
44 71				

Pattern 10

11.92	9.22	6.90	4.63	2.34
22.88	17.91	13.49	9.16	
32.14	25.66	19.71		
39.28	32 58			
44.68				

Pattern 11

11.89	9.18	6.84	4.60	2.32
22.84	17.81	13.42	9.06	
32.07	25.58	19.58		
39.24	32.47			
44.64				

Pattern 12

11.87	9.14	6.80	4.56	2.30
22.79	17.76	13.32	9.01	
32.03	25.48	19.50		
39.18	32 41			
44.62				

Pattern 13

11.84	9.11	6.76	4.53	2.28
22.76	17.68	13.27	8.94	
31.98	25.42	19.40		
39.16	32.33			
44.59				

Pattern 14

11.83	9.07	6.73	4.50	2.26
22.72	17.64	13.20	8.90	
31.95	25.35	19.34		
39.12	32.29			
44.58				

Table 101 (continued)

Pattern 15

11.81	9.05	6.69	4.47	2.25
22.70	17.58	13.15	8.85	
31.91	25.30	19.28		
39.10	32.24			
44.56				

Pattern 16

11.80	9.02	6.67	4.45	2.21
22.67	17.55	13.10	8.81	
31.89	25.25	19.22		
39.07	32.20			
44.55				

Pattern 17

11.78	9.00	6.64	4.43	2.22
22.66	17.51	13.06	8.78	
32.86	25.22	19.18		
39.05	32.16			
44.54				

of $u_{i5}^{(0)}$ we take

$$u_{i5}^{(0)} = \frac{15.45}{6}(6-i) \quad (i=1, 2, 3, 4, 5),$$

i.e. $u_{15}^{(0)} = 12.88$, $u_{25}^{(0)} = 10.30$, $u_{35}^{(0)} = 7.72$, $u_{45}^{(0)} = 5.15$, $u_{55}^{(0)} = 2.58$. Proceed analogously with the right-hand column putting $u_{5j}^{(0)} = u_{i5}^{(0)}$.

Then consider the next row. We assume that the function $u(x, y)$ decreases linearly from 29.39 to 5.15. Reasoning in the same way as in the preceding case, we obtain the values $u_{i4}^{(0)}$ $(i=2, 3, 4, 5)$, and hence u_{4j}^{0} $(j=1, 2, 3, 4)$. Continue the process until the entire table of the initial approximation is filled in (Table 101, Pattern 1).

2. Computing successive approximations by formula (1). Take Pattern 1, attach it to the basic pattern (Fig. 10) and compute successively by formula (1) for $k=1$

$$u_{15}^{(1)} = \frac{1}{4}(u_{25}^{(0)} + u_{05}^{(0)} + u_{16}^{(0)} + u_{14}^{(0)}) =$$

$$= \frac{1}{4}(10.30 + 15.45 + 0 + 24.54) = 12.57,$$

$$u_{14}^{(1)} = \frac{1}{4}(u_{24}^{(0)} + u_{04}^{(0)} + u_{15}^{(0)} + u_{13}^{(0)}) =$$

$$= \frac{1}{4}(19.69 + 29.39 + 34.05 + 12.88) = 24.00$$

and so on.

Enter all the results in Pattern 2 and find the next approximation $u_{ij}^{(2)}$ in a similar way. The computations are carried out until the values of two subsequent iterations differ by not more than 0.05. The results of the 16th and 17th iterations satisfy this condition. The patterns corresponding to the successive iterations are given in Table 101. Since the table is symmetric, the patterns are half filled in.

Example 2. Find the solution of the Laplace equation for a square under boundary conditions indicated in Fig. 11, taking $h=\frac{1}{6}$.

Solution.

1. *Computing the initial approximation.* To compute the initial approximation first construct a net with spacing $h=\frac{1}{3}$, denoting the values of the required function at the mesh points by a, b, c, d (Fig. 11). Note that from symmetry of the boundary conditions it follows that

$$a=b=c=d.$$

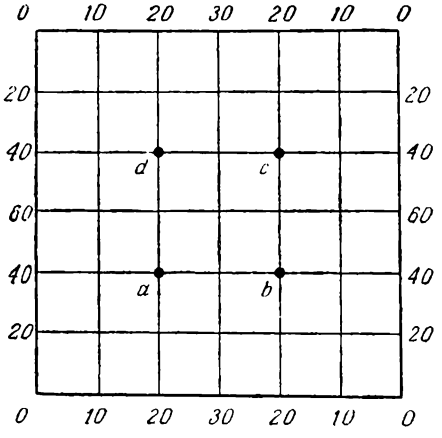

Fig. 11

Therefore it is sufficient to set up only one equation

$$40+b+20+d-4a=0.$$

Taking advantage (3), we get

$$a=30.$$

Then compute the initial approximation for $h=\frac{1}{6}$. To this end construct the pattern (Fig. 12), and enter in it the boundary conditions and the obtained values at the four points

$$u_{22}^{(0)}=u_{42}^{(0)}=u_{24}^{(0)}=u_{44}^{(0)}=30.$$

Using these quantities, find the values $u_{ij}^{(0)}$ at the remaining points of the net. Consider in detail the computation of $u_{i1}^{(0)}$ $(i=1, 2, 3, 4, 5)$.

6	0	10	20	30	20	10	0
5	20	22.5	24.4	25	24.4	22.5	20
4	40	35	30	30	30	35	40
3	60	40	35	30	35	40	60
2	40	35	30	30	30	35	40
1	20	22.5	24.4	25	24.4	22.5	20
0	0	10	20	30	20	10	0
j / i	0	1	2	3	4	5	6

Fig. 12

The values $u_{11}^{(0)}$, $u_{31}^{(0)}$ are obtained by formula (6) of Sec. 10.2:

$$u_{11}^{(0)} = \frac{1}{4}(u_{00} + u_{20} + u_{02} + u_{22}^{(0)}) = \frac{1}{4}(0 + 20 + 40 + 30) = 22.5,$$

$$u_{31}^{(0)} = \frac{1}{4}(u_{20} + u_{40} + u_{22}^{(0)} + u_{42}^{(0)}) = \frac{1}{4}(20 + 20 + 30 + 30) = 25.$$

The value $u_{21}^{(0)}$ is found by formula (5) of Sec. 10.2:

$$u_{21}^{(0)} = \frac{1}{4}(u_{20} + u_{22}^{(0)} + u_{11}^{(0)} + u_{31}^{(0)}) = 24.4.$$

By virtue of symmetry we put

$$u_{51}^{(0)} = u_{11}^{(0)}, \quad u_{41}^{(0)} = u_{21}^{(0)}.$$

The values $u_{ij}^{(0)}$ are computed analogously for $j = 2, 3, 4, 5$.

2. *Computing successive approximations.* Note that by virtue of symmetry it is sufficient to accomplish computations only for a quarter of the square. To accelerate convergence of the iteration proceed as follows. Find $u_{11}^{(1)}$ by formula (1) for $k = 1$:

$$u_{11}^{(1)} = \frac{1}{4}(u_{10} + u_{12}^{(0)} + u_{01} + u_{21}^{(0)}) = 22.3.$$

The obtained value is used for computing $u_{21}^{(1)}$, i.e.

$$u_{21}^{(1)} = \frac{1}{4}(u_{11}^{(1)} + u_{31}^{(0)} + u_{20} + u_{24}^{(0)}) = \frac{1}{4}(22.3 + 25 + 20 + 30) = 24.3.$$

In computing $u_{31}^{(1)}$ we use the values $u_{21}^{(1)} = u_{41}^{(1)}$, and so on. The process of iterations is continued untill the results of two conse-

quent approximations differ by not more than 0.1. The results of the successive approximations for a quarter of the square are presented in Table 102.

Table 102

Values of the Successive Approximations (Example 2)

40	33.1	30.6	29.6	30.6
60	40.3	32.9	31.2	32.9
40	33.1	30.6	29.6	30.6
20	22.3	24.3	27.2	24.3
0	10	20	30	20

40	33.2	30.2	29.9	30.2
60	39.8	32.8	31.3	32.8
40	33.2	30.2	29.9	30.2
20	21.9	24.9	27.4	24.9
0	10	20	30	20

40	33.1	30.2	29.9	30.2
60	39.8	32.8	31.3	32.8
40	33.1	30.2	29.9	30.2
20	22.0	24.9	27.4	24.9
0	10	20	30	20

Example 3. Table 103 gives the approximate solution of the Laplace equation for a unit square with the indicated boundary values for $h=0.1$. It is required to estimate the error of this solution by the Runge method.

Solution. Solve the problem anew with the spacing $2h=0.2$, taking the initial approximation from Table 103. The results of computations with the spacing $2h=0.2$ are listed in Table 104. Then find the differences $u_h - u_{2h}$ between the values of the required solution obtained with the spacings $h=0.1$ and $2h=0.2$, and compute the error ε_h by formula (2) (see Tables 105 and 106).

Problems

1. Applying the Liebmann method of averaging, approximate the solution of Laplace's equation with $h=\frac{1}{8}$ in a square with the vertices $A(0, 0)$, $B(0, 1)$, $C(1, 1)$, $D(1, 0)$. The boundary

Table 103

Approximate Solution of a Boundary Value Problem with $h=0.1$

	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	
6.75	5.86	5.06	4 34	3.68	3 04	2.42	1.82	1.21	0.60	0.00
13.38	11.60	10.04	8.63	7.32	6.06	4.84	3.63	2.42		0.00
19.70	17.12	14.86	12.81	10.89	9.04	7.23	5.43			0.00
25.60	22 34	19.47	16.85	14.38	11.98	9.60				0.00
30.95	27.17	23.82	20.73	17.78	14.88					0.00
35.66	31.56	27.89	24.46	21.12						0.00
39.67	35.50	31.72	28.09							0.00
42.98	39.05	35.38								0.00
45.63	42.34									0.00
	45.63	42.98	39.67	35.66	30.95	25.60	19.70	13.38	6.75	

conditions are given in Table 107. Carry out iterations accurate to 10^{-2}.

2. Approximate the solution of Laplace's equation in the square $ABCD$ for the boundary conditions indicated in Table 108 with the spacing $h=\frac{1}{6}$ for the following values of the parameters:

$$\alpha=0.9+0.1\cdot k, \quad k=0,\ 1,\ 2, \quad \beta=1.01+0.02\cdot n, \quad n=0,\ 1,\ 2.$$

Carry out iterations to within 10^{-2}.

3. Find approximate solutions of the Laplace equation for the domains and boundary conditions given in Fig. 13, *a* to *c*. Go on with the iterations until the difference between the subsequent values of the function for all points become less than 0.005.

Table 104

Approximate Solution of a Boundary Value Problem with $2h = 0.2$

	0.00	0.00	0.00	0.00	
13.38	10.05	7.32	11.84	2.42	0.00
25.60	19.50	14.40	9.62		0.00
35.66	27.95	21.17			0.00
42.98	35.46				0.00
	42.98	35.66	25.60	13.38	

Table 105

The Differences $u_h - u_{2h}$

0.01	0.00	0.00	0.00
0.03	0.02	0.02	
0.06	0.05		
0.08			

Table 106

Errors ε_n

	0.003		0.000		0.000		0.000	
	0.010		0.007		0.007			
	0.020		0.017					
	0.027							

Table 107

Boundary Conditions for Problem 1

Variant No.	$u \mid_{AB}$	$u \mid_{BC}$	$u \mid_{CD}$	$u \mid_{AD}$
1	$30y$	$30(1-x^2)$	0	0
2	$30y$	$30\cos\frac{\pi x}{2}$	$30\cos\frac{\pi y}{2}$	0
3	$50y(1-y^2)$	0	0	$50\sin \pi x$
4	$20y$	20	$20y^2$	$50x(1-x)$
5	0	$50x(1-x)$	$50y(1-y^2)$	$50x(1-x)$
6	$30\sin \pi y$	$20x$	$20y$	$30x(1-x)$
7	$30(1-y)$	$20\sqrt{x}$	$20y$	$30(1-x)$
8	$50\sin \pi y$	$30\sqrt{x}$	$30y^2$	$50\sin \pi x$
9	$40y^2$	40	40	$40\sin\frac{\pi x}{2}$
10	$50y$	$50(1-x)$	0	$60x,\ 0 \leqslant x < 1/2$ $60(1-x),\ 1/2 \leqslant x \leqslant 1$

Table 108

Boundary Conditions for Problem 2

$u \mid_{AD}$	0	17.28	29.05α	40.00	29.05β	17.28	4.31
$u \mid_{BC}$	0	9.81	17.98α	29.12	38.25β	42.31	50.00
$u \mid_{AB}$	0	0	0	0	0	0	0
$u \mid_{DC}$	4.31	6.98	12.38β	19.14	30.10α	40.16	50.00

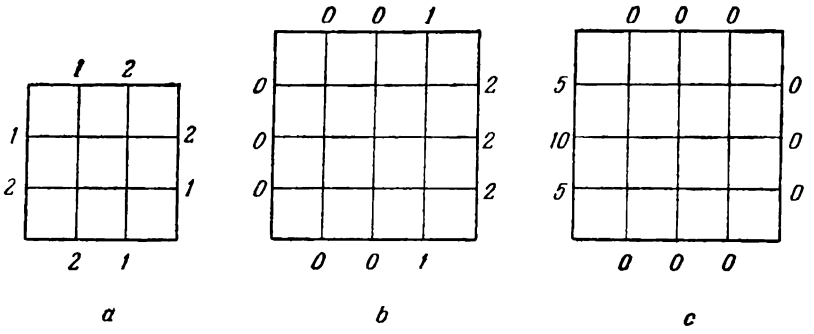

Fig. 13

4. Approximate the solution of Laplace's equation for a unit square with spacing $h=\frac{1}{8}$. Take the boundary conditions on the left-hand side of the square equal to 2.5; 5.0; 7.5, 10.0; 7.5; 5.0; 2.5; the rest of the boundary values are equal to zero. Carry out the iterations accurate to 10^{-4}. In choosing the first approximation use the solution of Problem 3, *c*.

10.4 SOLVING BOUNDARY VALUE PROBLEMS FOR CURVILINEAR DOMAINS

If the boundary Γ of a domain G is curvilinear, then the values of u_{ij} for the boundary points are obtained by transferring the values from the points of the boundary Γ. The error resulting from such a procedure can be considerably reduced by setting up an equation of the following form for each boundary point (see [2], [13], [35]):

(1) for the point A_h (Fig. 14)

$$u_{A_h}=\frac{\delta_1 u_B+hu_A}{\delta_1+h}; \tag{1}$$

(2) for C_h (Fig. 14)

$$u_{C_h}=\frac{\delta_2 u_D-hu_C}{\delta_2-h}. \tag{2}$$

Having obtained one of such equations for each boundary point, and adding them to system (4) or (5) of Sec. 10.2, we get a system of algebraic equations with respect to the values of u_{ij} at the mesh points. If this system is solved by the Liebmann method, then the successive approximations of the boundary values are computed

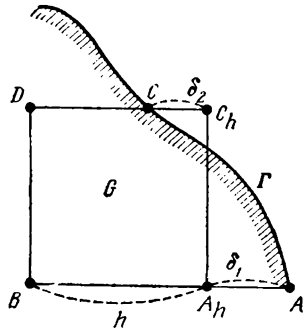

Fig. 14

by the formulas

$$u_{A_h}^{(k+1)} = u_A + \frac{u_B^{(k)} - u_A}{h + \delta_1}\,\delta_1, \tag{3}$$

$$u_{C_h}^{(k+1)} = u_C + \frac{u_D^{(k)} - u_C}{\delta_2 - h}\,\delta_2. \tag{4}$$

Example 1. Approximate the solution of the equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0, \tag{5}$$

satisfying on the circle $x^2 + y^2 = 16$ the condition

$$u|_\Gamma = x^2 y^2. \tag{6}$$

Solution. Since the solution is symmetric, let us consider a quarter of the circle.

Filling in the patterns:

(1) Take a large mesh net with spacing $h = 2$ (Fig. 15). $M(\sqrt{12}, 2)$ is the point of the boundary Γ nearest to the mesh point $A(4, 2)$ of the net, therefore we put

$$u(A) \approx u(M) = 12 \cdot 2^2 = 48.$$

Analogously, $M'(2, \sqrt{12})$ is the boundary point closest to the mesh point $A'(2, 4)$, therefore $u(A') \approx u(M') = 48$.

At the mesh points $C(4, 0)$ and $C'(0, 4)$ we obviously have

$$u(C) = u(C') = 0.$$

Denoting, for convenience, the values of the function $u(x, y)$ at the interior mesh points by a, b, c (Fig. 15), and taking into consideration the fact that the problem is symmetric, form a system of finite-difference equations

$$a = \frac{1}{4} \cdot 4b, \quad b = \frac{1}{4}(2c + a + 0), \quad c = \frac{1}{4}(48 + 48 + 2b).$$

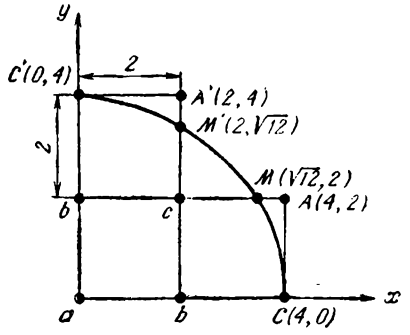

Fig. 15

From this system we find

$$a=24, \quad b=24, \quad c=36.$$

(2) Take a smaller-mesh net (Fig. 16) with spacing $h=1$ and boundary values not specified. Put

$$u(A)=u(A')=15, \quad u(B)=u(B')=48, \quad u(C)=63.$$

Using the values of the function $u(x, y)$ both at the points of the larger-mesh net with spacing $h=2$ and at the boundary points, and taking into consideration symmetry of the problem, form finite-difference equations for the values a, b, c, d, e, f (Fig. 16). For the values a, b, d, f we set up equations (5), and for c and e,

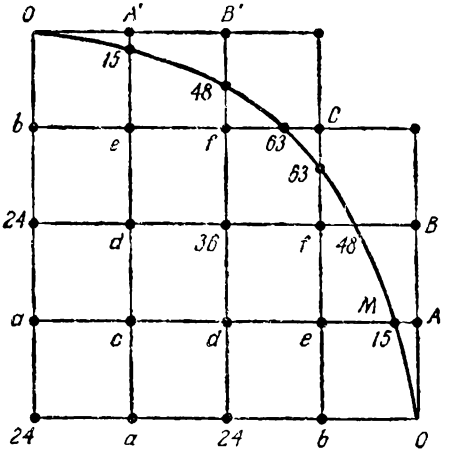

Fig. 16

equations (6) (see Sec. 10.2). Thus, we get

$$a=\frac{1}{4}(24+24+c+c), \quad f=\frac{1}{4}(48+e+63+36),$$

$$b=\frac{1}{4}(e+e+0+24), \quad c=\frac{1}{4}(24+24+24+36),$$

$$d=\frac{1}{4}(e+c+24+36), \quad e=\frac{1}{4}(0+36+48+24),$$

whence we approximately find

$$a=26, \quad b=20, \quad c=27, \quad d=28, \quad e=27, \quad f=44.$$

(3) Specify the values of $u(x, y)$ at boundary points using formula (4). For point A we have

$$\delta_A=|MA|=4-\sqrt{15}\approx 0.13,$$

hence

$$u_A^{(1)}=u_M+\frac{e-u_M}{\delta_A-h}\delta_A=15-\frac{1.56}{0.87}\approx 13.$$

Carry out similar computations for point B:

$$\delta_B=|NB|=4-\sqrt{12}\approx 0.6,$$

$$u_B^{(1)}=u_N+\frac{f-u_N}{\delta_B-h}\delta_B=48+\frac{0.4}{0.4}=49.$$

Thus, for the boundary points we get

$$u_A^{(1)}=u_A^{(2)}=13, \quad u_B^{(1)}=u_B^{(2)}=49, \quad u_C=73.$$

(4) Compile a table of the initial values (Pattern 1) and specify successively the values of the required function $u(x, y)$ at the interior points by formulas (1) of Sec. 10.3 till the values obtained in two successive iterations differ by not more than unity. The results of the computations are entered in Pattern 2 and 3 (Table 109). The approximate values of the required function in Patterns 2 and 3 differ by not more than unity. For the sake of comparison, Pattern 3a contains the values of the exact solution of the problem $u(x, y)=x^2y^2+\frac{1}{8}[256-(x^2+y^2)^2]$ at the mesh points.

Problems

Using the difference method with spacing h find the solution of Laplace's equation in domain G for the given boundary conditions. Solve the finite-difference system by the Liebmann method, specifying the boundary values.

Table 109

Values of the Successive Approximations for Problem (5), (6)

Pattern 1

0	13	49		
20	27	44	73	
24	28	36	44	49
26	27	28	27	13
24	26	24	20	0

Pattern 2

20	27	46	
26	29	37	46
26	27	29	27
26	26	26	20

Pattern 3

20	27	46	
26	30	38	46
26	28	30	27
26	26	26	20

Pattern 3a

0	12	46		
22	28	47	73	
30	33	40	47	46
32	32	33	28	12
32	32	30	22	0

1. Spacing $h=0.1$, domain G is bounded by the curves $2y=1-4x^2$, $y=0$, $x=0$, boundary conditions have the form

$$u|_{x=0}=0, \quad u|_{y=0}=(1-4x^2)x, \quad u|_{2y=1-4x^2}=12xy^2.$$

2. Spacing $h=0.2$, domain G is defined by the conditions

$$x^2+(y+3)^2\leqslant 16, \quad y\geqslant 0,$$

the boundary conditions have the form

(a) $u|_{y=0}=0$, $u|_C=2y(2x^2+3y)$, where C is a circle $x^2+(y+3)^2=16$,

(b) $u|_{y=0}=x(7-x^2)$, $u|_C=4xy^2$, where C is a circle $x^2+(y+3)^2=16$.

10.5 THE NET METHOD FOR THE PARABOLIC-TYPE EQUATION

Consider the *mixed boundary value problem for the heat conductivity equation*, namely: find the function $u(x, t)$ which satisfies the equation

$$\frac{\partial u}{\partial t}=a^2\frac{\partial^2 u}{\partial x^2}, \tag{1}$$

the initial condition

$$u(x, 0)=f(x) \quad (0<x<s) \tag{2}$$

and the boundary conditions

$$u(0, t)=\varphi(t), \quad u(s, t)=\psi(t). \tag{3}$$

Reduced, in particular, to problem (1)-(3) is the problem on heat propagation in a uniform bar of length s (see [48], [52]). By introducing the new variable $\tau=a^2t$ equation (1) is reduced to the

form

$$\frac{\partial u}{\partial \tau} = \frac{\partial^2 u}{\partial x^2},$$

therefore, henceforth we put $a = 1$.

Construct in the half-strip $t \geqslant 0$, $0 \leqslant x \leqslant s$ (Fig. 17) two families of parallel straight lines:

$$x = ih \quad (i = 0, 1, 2, \ldots), \quad t = jl \quad (j = 0, 1, 2, \ldots).$$

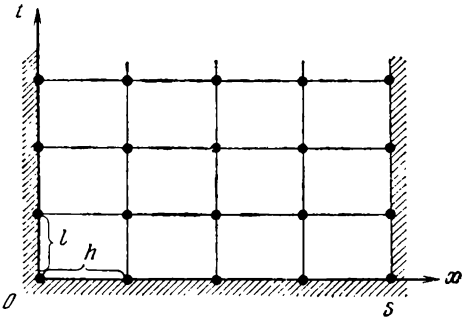

Fig. 17

Introduce the notation $x_i = ih$, $t_j = jl$, $u(x_i, t_j) = u_{ij}$ and in each interior point (x_i, t_j) replace the derivative $\frac{\partial^2 u}{\partial x^2}$ by the difference relation

$$\left(\frac{\partial^2 u}{\partial x^2}\right)_{ij} \approx \frac{u_{i+1,j} - 2u_{ij} + u_{i-1,j}}{h^2} \tag{4}$$

and the derivative $\frac{\partial u}{\partial t}$ by one of the two difference relations

$$\left(\frac{\partial u}{\partial t}\right)_{ij} \approx \frac{u_{i,j+1} - u_{ij}}{l}, \tag{5}$$

$$\left(\frac{\partial u}{\partial t}\right)_{ij} \approx \frac{u_{ij} - u_{i,j-1}}{l}. \tag{6}$$

Then for equation (1) at $a = 1$ we get two types of finite-difference equations:

$$\frac{u_{i,j+1} - u_{ij}}{l} = \frac{u_{i+1,j} - 2u_{ij} + u_{i-1,j}}{h^2}, \tag{7}$$

$$\frac{u_{ij} - u_{i,j-1}}{l} = \frac{u_{i+1,j} - 2u_{ij} + u_{i-1,j}}{h^2}. \tag{8}$$

Denoting $\sigma = l/h^2$, reduce these equations to the form

$$u_{i,j+1} = (1 - 2\sigma)\, u_{ij} + \sigma (u_{i+1,j} + u_{i-1,j}), \tag{9}$$

$$(1 + 2\sigma)\, u_{ij} - \sigma (u_{i+1,j} + u_{i-1,j}) - u_{i,j-1} = 0. \tag{10}$$

Note that for setting up the equation (7) we used the *explicit diagram* of mesh points shown in Fig. 18, and for equation (8), the *implicit diagram* given in Fig. 19.

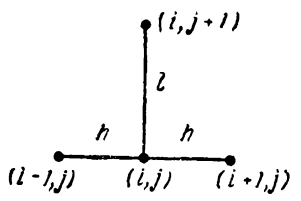

Fig. 18

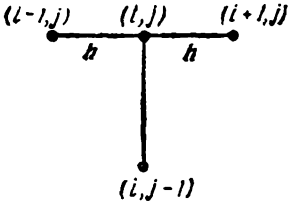

Fig. 19

When choosing the number σ in equations (9), (10) the following two circumstances should be taken into consideration:

(1) the error due to the replacement of a differential equation by a difference equation must be of the least value;

(2) the difference equation must be stable.

It is proved (see [2], [13]) that equation (9) will be stable for $0<\sigma\leqslant\frac{1}{2}$, and the equation (10), for any σ. Equation (9) has the most convenient form for $\sigma=\frac{1}{2}$:

$$u_{i,j+1}=\frac{u_{i-1,j}+u_{i+1,j}}{2}, \tag{11}$$

and for $\sigma=1$ 6:

$$u_{i,j+1}=\frac{1}{6}(u_{i-1,j}+4u_{ij}+u_{i+1,j}). \tag{12}$$

The error estimates of the approximate solutions obtained from equations (11), (12), (10) in the strip $0\leqslant x\leqslant s$, $0\leqslant t\leqslant T$ have, respectively, the form (see [31])

$$|u-\tilde{u}|\leqslant\frac{T}{3}M_1h^2, \tag{13}$$

$$|u-\tilde{u}|\leqslant\frac{T}{135}M_2h^4, \tag{14}$$

$$|u-\tilde{u}|\leqslant T\left(\frac{l}{2}+\frac{h^2}{12}\right)M_1, \tag{15}$$

where u is the exact solution of problem (1)-(3),

$M_1=\max\{|f^{(0)}(x)|,\ |\varphi''(t)|,\ |\psi''(t)|\}$ for $0\leqslant t\leqslant T$, $0\leqslant x\leqslant s$,

$M_2=\max\{|f^{(0)}(x)|,\ |\varphi^{(4)}(t)|,\ |\psi^{(4)}(t)|\}$ for $0\leqslant t\leqslant T$, $0\leqslant x\leqslant s$.

As is seen from the above error estimates, equation (12) yields a higher accuracy of the solution as compared with equation (11).

But the latter equation has a simpler form and, besides, the spacing l with respect to the argument t for the equation (12) must be considerably smaller, which leads to a larger amount of computations. The equation (10) gives a lower accuracy, but l and h in this case are chosen independently of each other. Equations (11) and (12) enable us to evaluate the function $u(x, y)$ for each layer by explicit formulas in terms of the values for the previous layer, whereas equation (10) (implicit diagram) does not possess this property.

The net method can be applied to solve a mixed boundary value problem for the non-homogeneous parabolic equation

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + F(x, t).$$

Then the corresponding difference equation, using the explicit diagram of points, has the form

$$u_{i, j+1} = (1-2\sigma)\, u_{ij} + \sigma (u_{i+1, j} + u_{i-1, j}) + lF_{ij}. \tag{16}$$

Whence we get for $\sigma = 1/2$

$$u_{i, j+1} = \frac{1}{2}(u_{i+1, j} + u_{i-1, j} + lF_{ij}, \tag{17}$$

and for $\sigma = 1/6$

$$u_{i, j+1} = \frac{1}{6}(u_{i-1, j} + 4u_{ij} + u_{i+1, j}) + lF_{ij}. \tag{18}$$

In this case we have the following error estimates (see [31]): for equation (17)

$$|\tilde{u} - u| \leqslant \frac{T}{4}\left(M_2 + \frac{1}{3} M_4\right) h^2,$$

for equation (18)

$$|\tilde{u} - u| \leqslant \frac{T}{72}\left(\frac{1}{3} M_3 + \frac{1}{5} M_6\right) h^4,$$

where

$$M_2 = \max\left|\frac{\partial^2 u}{\partial t^2}\right|, \quad M_3 = \max \frac{\partial^3 u}{\partial t^3}, \quad M_4 = \max \frac{\partial^4 u}{\partial x^4}, \quad M_6 = \max \frac{\partial^6 u}{\partial x^6}.$$

Example 1. Using the difference equation (11), approximate the solution of the equation

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}, \tag{19}$$

satisfying the following conditions

$$u(x, 0) = \sin \pi x \ (0 \leqslant x \leqslant 1), \text{ and } (0, t) = u(1, t) = 0 \quad (0 \leqslant t \leqslant 0.025). \tag{20}$$

Solution. Choose the spacing $h=0.1$ for the argument x. Since $\sigma=\frac{1}{2}$, we get the spacing $l=\frac{h^2}{2}=0.005$ with respect to the argument t. Enter the initial and boundary values in Table 110. Taking into account their symmetry, fill in the table only for $x=0;\ 0.1;\ 0.2;\ 0.3;\ 0.4;\ 0.5$. The values of the function $u(x, t)$ on the first layer are found using the values on the initial layer and the boundary conditions by formula (11) for $j=0$.

$$u_{i1}=\frac{u_{i+1,\,0}+u_{i-1,\,0}}{2}.$$

Thus, we get

$$u_{11}=\frac{1}{2}(u_{20}+u_{00})=\frac{1}{2}(0.5878+0)=0.2939,$$

$$u_{21}=\frac{1}{2}(u_{30}+u_{10})=\frac{1}{2}(0.8090+0.3090)=0.5590$$

and so on.

Enter the obtained values of u_{i1} $(i=1, 2, 3, 4, 5)$ in the second row of Table 110. Then pass to computing the values on the second layer by formula (11) for $j=1$:

$$u_{i2}=\frac{u_{i+1,\,1}+u_{i-1,\,1}}{2}.$$

We determine successively the values of u_{ij} for $t=0.005;\ 0.010;\ 0.015;\ 0.020;\ 0.025$ in a similar way. The last two rows of the table present the values of the exact solution of the problem $\tilde{u}(x, t)=e^{-\pi^2 t}\sin\pi x$ and the modulus of the difference $|\tilde{u}-u|$ at $t=0.025$.

Table 110

Solution of Problem (19), (20) by Formula (11)

j	t \ x	0	0.1	0.2	0.3	0.4	0.5
0	0	0	0.3090	0.5878	0.8090	0.9511	1.0000
1	0.005	0	0.2939	0.5590	0.7699	0.9045	0.9511
2	0.010	0	0.3795	0.5316	0.7318	0.8602	0.9045
3	0.015	0	0.2658	0.5056	0.6959	0.8182	0.8602
4	0.020	0	0.2528	0.4808	0.6619	0.7780	0.8182
5	0.025	0	0.2404	0.4574	0.6294	0.7400	0.7780
$\tilde{u}(x, t)$	0.025	0	0.2414	0.4593	0.6321	0.7431	0.7813
$\lvert\tilde{u}-u\rvert$	0.025	0	0.0010	0.0019	0.0027	0.0031	0.0033

For the sake of comparison let us give the error estimate obtained by formula (13). For the given problem $\varphi(t)=\psi(t)=0$, $f^{(4)}(x)=\pi^4\sin\pi x$, hence, $M_1=\pi^4$. Thus, we get

$$|\tilde{u}-u|\leqslant\frac{0.025}{3}\pi^4h^2=\frac{0.025}{3}\cdot 97.22\cdot 0.01=0.0081.$$

Example 2. Using the difference equation (12), find the solution of problem (19), (20) for $0\leqslant t\leqslant 0.01$. Estimate the error of the obtained solution.

Solution. Take the spacing $h=0.1$ with respect to the argument x. Since for the formula (12) $\sigma=\frac{1}{6}$, we get for the argument t spacing $l=\frac{0.01}{6}\approx 0.0017$. Enter the initial and boundary values in Table 111.

By virtue of symmetry of the solution it suffices to fill in the table for $0\leqslant x\leqslant 0.5$. Then pass to compute by formula (12). For the first layer at $j=1$ we get

$$u_{i1}=\frac{1}{6}(u_{00}+4u_{10}+u_{20}),$$

Table 111

Solution of Problem (19), (20) by Formula (12)

j	t \ x	0	0.1	0.2	0.3	0.4	0.5
0	0	0	0.309017	0.587785	0.809017	0.951057	0.000000
1	0.0017	0	0.303976	0.578196	0.795818	0.935541	0.983686
2	0.0033	0	9.299017	0.568763	0.782835	0.920278	0.967638
3	0.0050	0	0.294138	0.559484	0.770063	0.905264	0.951852
4	0.0067	0	0.289339	0.550356	0.757500	0.890495	0.936322
5	0.0083	0	0.284619	0.541377	0.745142	0.875967	0.921046
6	0.0100	0	0.279976	0.532545	0.732982	0.861676	0.906018
$\tilde{u}(x,\ t)$	0.01	0	0.279975	0.532544	0.732984	0.861675	0.906018

whence we find successively

$$u_{11}=\frac{1}{6}(0+4\cdot 0.309017+0.587785)=0.303976,$$

$$u_{21}=\frac{1}{6}(0.309017+4\cdot 0.587785+0.809017)=0.578196,$$

. . . . . .

$$u_{51}=\frac{1}{6}(0.951057=4\cdot 1+0.951057)=0.983686.$$

Computations for the subsequent layers are carried out in a similar way. For the error estimate by formula (14) at $t=0.01$ we have $\varphi(t)=\psi(t)=0$, $f^{(6)}x=\pi^6\sin\pi x$, $M_2=\pi^6$. Thus

$$|u-\tilde{u}|\leqslant\frac{0.01}{135}\pi^6h^4\approx\frac{0.01}{135}\,958.6\cdot 10^{-4}\approx 7.10^{-6}.$$

Given in the last row of Table 111 are the values of the exact solution of $\tilde{u}=e^{-\pi^2 t}\sin x$ for $t=0.01$. The comparison shows that the error of the obtained solution does not exceed $2\cdot 10^{-6}$.

Problems

Approximate the solution of the equation

$$\frac{\partial u}{\partial t}=\frac{\partial^2 u}{\partial x^2},$$

satisfying the conditions

$$u(x,\,0)=f(x),\quad u(0,\,t)=\varphi(t),\quad u(1,\,t)=\psi(t),$$

for the values $0\leqslant t\leqslant T$, taking the spacing $h=0.1$ along x. In Problems 1 to 3 make use of the difference equation (11), in Problems 4 to 6, difference equation (12).

1. $f(x)=(ax^2+b)\sin\pi x$, $\varphi(t)=\psi(t)=0$, $T=0.02$, $a=1.1;\ 1.3;\ 1.5$, $b=1.1+0.1\cdot n$, $n=0,\ 1,\ 2,\ 3,\ 4$.

2. $f(x)=e^{-bx}\sin ax$, $\varphi(t)=0$, $\psi(t)=e^{-b}\sin a$, $T=0.02$, $a=\pi/12,\ \pi/4,\ \pi/3$, $b=0.1\cdot k$, $k=1,\ 2,\ \ldots,\ 5$.

3. $\varphi(t)=\psi(t)=0$, $f(x)$ is given by the table

x	0	0.1	0.2	0.3	0.4	0.5
$f(x)$	0	0.0196	$0.0431+a$	0.0742	0.1116	$0.1537+2a$
x		0.6	0.7	0.8	0.9	1.0
$f(x)$		0.1994	0.1256	$0.0614-a$	0.0031	0

$T=0.02$, $a=0.02\cdot n$, $n=0, 1, 2, \ldots, 5$.

4. $f(x)=(ax^2+b)e^{-x}$, $\varphi(t)=b$, $\psi(t)=(a+b)e^{-1}$, $T=0.01$, $a=1.1$; 1.3; 1.5, $b=2.1+0.1\cdot n$, $n=0, 1, 2, 3, 4$.

5. $f(x)=x(1-x)(ax^4+b)$, $\varphi(t)=\psi(t)=0$, $T=0.01$, $a=0.5$; 0.7; 0.9, $b=0.5+0.1\cdot n$, $n=0, 1, 2, 3, 4$.

6. $\varphi(t)=0$, $\psi(t)=f(1)$, $f(x)$ is specified by the table

x	0	0.1	0.2	0.3	0.4
$f(x)$	0	0.0221	$0.0425+a$	0.1008	0.1545

x	0.5	0.6	0.7	0.8	0.9	1.0
$f(x)$	$0.1721+2a$	0.2032	0.2895	$0.3581-a$	0.4010	0.4500

$T=0.01$, $a=0.02\cdot n$, $n=0, 1, 2, 3, 4, 5$.

7. Approximate the solution of the equation

$$\frac{\partial u}{\partial t}=\frac{\partial u^2}{\partial x^2}+2x+t$$

with the spacing $h=0.1$ along x, satisfying the initial and boundary conditions of Problem 1. Take advantage of difference equation (17).

8. Approximate the solution of the equation

$$\frac{\partial u}{\partial t}=\frac{\partial^2 u}{\partial x^2}+3t\sin x$$

with $h=0.1$ along x, satisfying the initial and boundary conditions of Problem 2. Take advantage of difference equation (18).

10.6 The "Passage" Method for Heat Conductivity Equation

Let it be required in the strip $0\leqslant x\leqslant a$, $0\leqslant t\leqslant T$ to find the solution of the equation

$$\frac{\partial u}{\partial t}=\frac{\partial^2 u}{\partial x^2}, \tag{1}$$

satisfying the following conditions

$$u(x, 0)=f(x), \quad u(0, t)=\varphi(t), \quad u(a, t)=\psi(t). \tag{2}$$

Choose the spacings h and l for x and t, respectively, replace the derivatives by finite-difference relations (4), (6) of Sec. 10.5 in each interior mesh point, compute the values of the functions $f(x)$,

$\varphi(t)$, $\psi(t)$ at the boundary points and, denoting $s=\frac{h^2}{l}$, we get the system

$$u_{i-1,\,j+1}-(2+s)\,u_{i,\,j+1}+u_{i+1,\,j+1}+su_{ij}=0$$
$$(i=1,\ 2,\ \ldots,\ n;\ j=0,\ 1,\ 2,\ \ldots), \tag{3}$$
$$u_{i0}=f(x_i), \tag{4}$$
$$u_{0j}=\varphi(t_j), \tag{5}$$
$$u_{nj}=\psi(t_j). \tag{6}$$

The "passage" method (see [2], [13], [31]) of solving system (3) to (6) consists in that equations (3) are reduced to the form

$$u_{i,\,j+1}=a_{i,\,j+1}(b_{i,\,j+1}+u_{i+1,\,j+1}), \tag{7}$$

where the numbers $a_{i,\,j+1}$, $b_{i,\,j+1}$ are determined in succession by the formulas

$$a_{1,\,j+1}=\frac{1}{2+s},\quad b_{1,\,j+1}=\varphi(t_{j+1})+su_{1j}, \tag{8}$$

$$\left.\begin{aligned} a_{i,\,j+1}&=\frac{1}{2+s-a_{i-1,\,j+1}},\\ b_{i,\,j+1}&=a_{i-1,\,j+1}b_{i-1,\,j+1}+su_{ij}\\ &(i=2,\ 3,\ \ldots,\ n). \end{aligned}\right\} \tag{9}$$

Then from the boundary condition (6) we find

$$u_{n,\,j+1}=\psi(t_{j+1})$$

and successively determine the values of $u_{i,\,j+1}$ by formula (7).

Thus, the "passage" method enables us to determine the values of the function $u(x,\ t)$ on the layer $t=t_{j+1}$, if its values on the layer $t=t_j$ are known.

Solving the problem:

Direct procedure. Using boundary conditions (5), find the numbers $a_{1,\,j+1}$, $b_{1,\,j+1}$, $a_{i,\,j+1}$, $b_{i,\,j+1}$ $(i=2,\ 3,\ \ldots,\ n)$ by formulas (8), (9).

Reverse procedure. From boundary condition (6) we get $u_{n,\,j+1}=\psi(t_{j+1})$. Then by formula (7) we compute

$$\left.\begin{aligned} u_{n-1,\,j+1}&=(u_{n,\,j+1}+b_{n-1,\,j+1})\,a_{n-1,\,j+1},\\ u_{n-2,\,j+1}&=(u_{n-1,\,j+1}+b_{n-2,\,j+1})\,a_{n-2,\,j+1},\\ &\ldots\ldots\ldots\ldots\ldots\ldots\ldots\\ u_{1,\,j+1}&=(u_{2,\,j+1}+b_{1,\,j+1})\,a_{1,\,j+1}. \end{aligned}\right\} \tag{10}$$

Example 1. Using the "passage" method, find the solution of the equation

$$\frac{\partial u}{\partial t}=\frac{\partial^2 u}{\partial x^2}, \tag{11}$$

satisfying the conditions

$$u(x, 0) = 4x(1-x), \quad u(0, t) = u(1, t) = 0. \tag{12}$$

Solution. Take $h = 0.1$, $l = 0.01$, hence $s = \frac{h^2}{l} = 1$. Using the "passage" method, find the values of the function $u(x, t)$ on the layer $t = 0.01$.

Table 112

Solution of Problem (11), (12) by the "Passage" Method

i	0	1	2	3	4	5	6	7	8	9	10
u_{i0}	0	0.360	0.640	0.840	0.960	1.000	0.960	0.840	0.640	0.360	0
u_{i0}		0.333	0.375	0.381	0.382	0.382	0.382	0.382	0.382	0.382	
b_{i1}		0.360	0.760	1.125	1.389	1.530	1.544	1.430	1.186	0.813	
u_{i1}	0	0.310	0.572	0.764	0.882	0.921	0.882	0.764	0.571	0.310	0

Direct procedure. Enter in the row u_{i0} of Table 112 the values of the initial function $f(x_i)$ $(i = 0, 1, \ldots, 10)$, by formulas (8) find the following numbers for $j = 0$:

$$a_{11} = \frac{1}{3}, \quad b_{11} = u_{10} = 0.36.$$

Then we compute successively by formulas (9) for $j = 0$

$$a_{21} = \frac{1}{3 - a_{11}} = \frac{3}{8} = 0.375, \quad b_{21} = a_{11} b_{11} + u_{20} = 0.12 + 0.64 = 0.760,$$

$$a_{31} = \frac{1}{3 - a_{21}} = \frac{1}{2.625} = 0.381, \quad b_{31} = a_{21} b_{21} + u_{30} =$$

$$= 0.375 \cdot 0.760 + 0.84 = 1.125$$

and so on.

Enter the obtained results in the rows a_{i1}, b_{i1} of Table 112.

Reverse procedure. From the boundary conditions we get

$$u_{10,\,1} = 0.$$

The values of u_{i1} $(i = 9, 8, \ldots, 1)$ are computed by formulas (10)

for $j=0$:

$$u_{91}=(u_{10,1}+b_{91})\,a_{91}=0.813\cdot 0.382=0.310,$$
$$u_{81}=(u_{91}+b_{81})\,a_{81}=(0.310+1.186)\cdot 0.382=0.571,$$
$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$
$$u_{11}=(u_{21}+b_{11})\,a_{11}=(0.572+0.360)\cdot 0.333=0.310.$$

Problems

Solve Problems 1 to 3 from Sec. 10.5 by the "passage" method and compare the results.

10.7 THE NET METHOD FOR A HYPERBOLIC-TYPE EQUATION

Consider a *mixed boundary value problem for the equation of a vibrating string*, which consists in finding the function satisfying the equation

$$\frac{\partial^2 u}{\partial t^2}=a^2\frac{\partial^2 u}{\partial x^2}, \tag{1}$$

and also the initial conditions

$$u(x,\,0)=f(x),\quad u_t(x,\,0)=\Phi(x)\quad (0\leqslant x\leqslant s) \tag{2}$$

and the boundary conditions

$$u(0,\,t)=\varphi(t),\quad u(s,\,t)=\psi(t). \tag{3}$$

Since introduction of the variable $\tau=at$ brings equation (1) to the form

$$\frac{\partial^2 u}{\partial t^2}=\frac{\partial^2 u}{\partial x^2}, \tag{4}$$

we may take $a=1$.

Constructing in $t\geqslant 0$, $0\leqslant x\leqslant s$ (Fig. 20) two families of parallel straight lines

$$x=ih\quad (i=0,\,1,\,2,\,\ldots,\,n),$$
$$t=jl\quad (j=0,\,1,\,2,\,\ldots),$$

replace the derivatives in equation (4) by difference relations. Taking advantage of symmetric formulas for derivatives, we have

$$\frac{u_{i,\,j+1}-2u_{ij}+u_{i,\,j-1}}{l^2}=\frac{u_{i+1,\,j}-2u_{ij}+u_{i-1,\,j}}{h^2}. \tag{5}$$

Denoting $\alpha=l/h$, we get the difference equation

$$u_{i,\,j+1}=2u_{ij}-u_{i,\,j-1}+\alpha^2(u_{i+1,\,j}-2u_{ij}+u_{i-1,\,j}). \tag{6}$$

It is proved (see [2]), that for $\alpha\leqslant 1$ this equation is *stable*.

In particular, at $\alpha = 1$ the equation (6) has the simplest form:

$$u_{i,\,j+1} = u_{i+1,\,j} + u_{i-1,\,j} - u_{j,\,j-1}. \tag{7}$$

Error estimate (see [2]) of the approximate solution, obtained from equation (6) $0 \leqslant x \leqslant s$, $0 < t \leqslant T$, has the form

$$|\tilde{u} - u| \leqslant \frac{h^2}{12}[(M_4 h + 2M_3)T + T^2 M_4], \tag{8}$$

where $\tilde{u}$ is the exact solution, $M_k = \max\left\{\left|\frac{\partial^k u}{\partial t^k}\right|, \left|\frac{\partial^k u}{\partial x^k}\right|\right\}$ $(k = 3, 4)$.

Not that for obtaining equation (6) we use the mesh points shown in Fig. 20. This is an explicit diagram since equation (6)

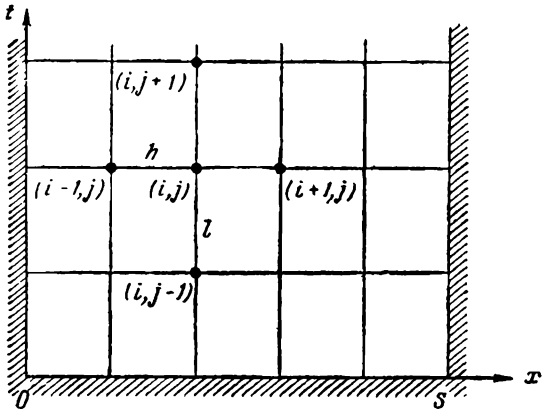

Fig. 20

makes it possible to find the values of the function $u(x, t)$ on the layer t_{j+1} if the values on the two previous layers are known. To approximate the solution of the problem (1) to (3), one must know the values of the solution on two initial layers. They can be found from the initial conditions by one of the following methods.

First method. In the initial condition (2) replace the derivative $u_t(x, 0)$ by the difference relation

$$\frac{u_{i1} - u_{j0}}{l} = \Phi(x_i) = \Phi_i;$$

for determining the values of $u(x, t)$ on the layers $j = 0$, $j = 1$, we get

$$u_{i0} = f_i, \quad u_{i1} = f_i + l\Phi_i. \tag{9}$$

In this case the error estimate of the values of u_{i1} has the form (see [2])

$$|\tilde{u}_{i1} - u_{i1}| \leqslant \frac{\alpha h}{2} M_2, \tag{10}$$

where

$$M_2 = \max\left\{\left|\frac{\partial^2 u}{\partial t^2}\right|, \left|\frac{\partial^2 u}{\partial x^2}\right|\right\}.$$

Second method. Replace the derivative $u_t(x, 0)$ by the difference relation $\frac{u_{i1}-u_{i,-1}}{2l}$, where $u_{i,-1}$ are the values of the function $u(x, t)$ on the layer $j=-1$. Then from the initial conditions (2) we have

$$u_{i0} = f_i, \quad \frac{u_{i1}-u_{i,-1}}{2l} = \Phi_i. \tag{11}$$

Write the difference equation (7) for the layer $j=0$:

$$u_{i1} = u_{i+1,0} + u_{i-1,0} - u_{i,-1}. \tag{12}$$

Eliminating from equations (11), (12) the values $u_{i,-1}$, we get

$$u_{i0} = f_i, \quad u_{i1} = \frac{1}{2}(f_{i+1} + f_{i-1}) + l\Phi_i. \tag{13}$$

The error estimate for the values u_{i1} has the form (see [2])

$$|\tilde{u}_{i1} - u_{i1}| \leqslant \frac{h^4}{12} M_4 + \frac{h^3}{6} M_3, \tag{14}$$

where

$$M_k = \max\left\{\left|\frac{\partial^k u}{\partial t^k}\right| \quad \left|\frac{\partial^k u}{\partial x^k}\right|\right\} \quad (k=3, 4).$$

This method of computing the initial values is considered in Example 1.

Third method. If a function $f(x)$ has a finite second derivative, then the values of u_{i1} can be determined with the aid of Taylor's formula

$$u_{i1} \approx u_{i0} + l\frac{\partial u_{i0}}{\partial t} + \frac{l^2}{2}\frac{\partial^2 u_{i0}}{\partial t^2}. \tag{15}$$

Using equation (4) and initial conditions (2), we can write

$$u_{i0} = f_i, \quad \frac{\partial u_{i0}}{\partial t} = \Phi_i, \quad \frac{\partial^2 u_{i0}}{\partial t^2} = \frac{\partial^2 u_{i0}}{\partial x^2} = f_i''.$$

Then, by formula (15), we have

$$u_{i1} \approx f_i + l\Phi_i + \frac{l^2}{2} f_i''. \tag{16}$$

The error of the values of u_{i1}, obtained by this formula, has the order $0(l^3)$.

Such method of computing the initial values is considered in Example 2.

Note. The net method is used to solve a mixed boundary value problem for the non-homogeneous equation

$$\frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} = F(x, t).$$

In this case the difference equation has the form

$$u_{i,j+1} = 2u_{ij} - u_{i,j-1} + \alpha^2 (u_{i+1,j} - 2u_{ij} + u_{i-1,j}) + \alpha^2 h^2 F_{ij}.$$

Example 1. Using the net method, find the solution of the problem

$$\left.\begin{aligned} &\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2}, \\ &u(x, 0) = 0.2x(1-x)\sin \pi x, \quad u_t(x, 0) = 0, \\ &u(0, t) = u(1, t) = 0. \end{aligned}\right\} \tag{17}$$

Solution. Take a square net with spacing $h = l = 0.05$. Find the values of $u(x, t)$ on the two initial layers by the second method. Taking into account that $\Phi(x) = 0$ and $f(x) = 0.2x(1-x)\sin \pi x$, we have

$$\left.\begin{aligned} &u_{i0} = f_i, \\ &u_{i1} = \frac{1}{2}(f_{i+1} + f_{i-1}) \\ &(i = 0, 1, 2, \ldots, 10). \end{aligned}\right\} \tag{18}$$

Filling in the table:

(1) Compute the values of $u_{i0} = f(x_i)$ for $x_i = ih$ and enter them in the first row of Table 113 (which corresponds to the value $t_0 = 0$). Since the problem is symmetric, fill in the table for $0 \leqslant x \leqslant 0.5$. The boundary values are written down in the first column which corresponds to the value $x_0 = 0$.

(2) Find u_{i1} by formula (18), using the values u_{i0} from the first row, entering the results in the second row of Table 113.

(3) Compute the values of u_{ij} on the subsequent layers by formula (7). For $j = 1$ we successively get

$$u_{12} = u_{21} + u_{01} - u_{10} = 0.0065 + 0 - 0.0015 = 0.0050,$$
$$u_{22} = u_{31} + u_{11} - u_{20} = 0.0122 + 0.0028 - 0.0056 = 0.0094,$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$u_{10,2} = u_{11,1} + u_{91} - u_{10,0} = 0.0478 + 0.0478 - 0.0500 = 0.0456.$$

For $j = 2, 3, \ldots, 10$ the computations are carried out analogously. The last row of the table contains the values of the exact solution at $t = 0.5$.

Example 2. Using the net method, find the solution of the problem

$$\left.\begin{aligned} &\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2}, \\ &u(x, 0) = x(\pi - x), \quad u_t(x, 0) = 0, \\ &u(0, t) = u(\pi, t) = 0. \end{aligned}\right\} \tag{19}$$

Table 113

Solution of Problem (17)

t_j \ x_i	0	0.05	0.10	0.15	0.20	0.25
0	0	0.0015	0.0056	0.0116	0.0188	0.0265
0.05	0	0.0028	0.0065	0.0122	0.0190	0.0264
0.10	0	0.0050	0.0094	0.0139	0.0198	0.0260
0.15	0	0.0066	0.0124	0.0170	0.0209	0.0256
0.20	0	0.0074	0.0142	0.0194	0.0228	0.0251
0.25	0	0.0076	0.0144	0.0200	0.0236	0.0249
0.30	0	0.0070	0.0134	0.0186	0.0221	0.0236
0.35	0	0.0058	0.0112	0.0155	0.0186	0.0199
0.40	0	0.0042	0.0079	0.0112	0.0133	0.0144
0.45	0	0.0021	0.0042	0.0057	0.0070	0.0074
0.50	0	−0.0001	−0.0001	0.0000	−0.0002	0.0000
$\tilde{u}(x_i, 0.5)$	0	0	0	0	0	0

t_j \ x_i	0.30	0.35	0.40	0.45	0.50
0	0.0340	0.0405	0.0457	0.0489	0.0500
0.05	0.0335	0.0398	0.0447	0.0478	0.0489
0.10	0.0322	0.0377	0.0419	0.0447	0.0456
0.15	0.0302	0.0343	0.0377	0.0397	0.0405
0.20	0.0277	0.0302	0.0321	0.0335	0.0338
0.25	0.0251	0.0255	0.0260	0.0262	0.0265
0.30	0.0227	0.0209	0.0196	0.0190	0.0186
0.35	0.0194	0.0168	0.0139	0.0120	0.0115
0.40	0.0140	0.0124	0.0092	0.0064	0.0054
0.45	0.0074	0.0064	0.0042	0.0026	0.0013
0.50	−0.0002	−0.0001	−0.0002	−0.0002	−0.0002
$\tilde{u}(x_i, 0.5)$	0	0	0	0	0

Solution. Take the spacing $h = l = \frac{\pi}{18}$. Find the values of $u(x, t)$ on the first two layers by the third method, using Taylor's formula.

Filling in the table:

(1) Compute the values $u_{i0} = f_i = x_i(\pi - x_i)$ $(i = 0, 1, \ldots, 18)$ and enter them in the first row of Table 114. Since the problem is symmetric, fill in the table for $0 \leqslant x \leqslant \frac{\pi}{2}$. Write down the boundary values in the first column of the table.

Table 114

Solution of Problem (19)

j	t_j \ x_i	0	$\pi/18$	$\pi/9$	$\pi/6$	$2\pi/9$	$5\pi/18$	$\pi/3$	$7\pi/18$	$4\pi/9$	$\pi/2$
0	0	0	0.518	0.975	1.371	1.706	1.980	2.193	2.346	2.437	2.467
1	h	0	0.487	0.944	1.340	1.675	1.950	2.163	2.315	2.406	2.437
2	$2h$	0	0.426	0.853	1.249	1.584	1.858	2.071	2.224	2.315	2.346
3	$3h$	0	0.366	0.731	1.097	1.432	1.706	1.919	2.071	2.163	2.193
4	$4h$	0	0.305	0.609	0.914	1.218	1.493	1.706	1.858	1.950	1.980
5	$5h$	0	0.244	0.487	0.731	0.975	1.218	1.432	1.584	1.675	1.706

(2) Determine u_{i1}. In our problem $\Phi_i = 0$, $f_i'' = -2$. Thus, by formula (16) we have

$$u_{i1} = u_{i0} - h^2 = u_{i0} - 0.03048,$$

whence we find the values of u_{i1} and enter them in the second row of the table.

(3) Calculate the values of $u_{i,\,j+1}$ for $j = 1, 2, 3, 4, 5$ by formula (7). For $j = 2$ we get

$$u_{12} = u_{21} + u_{01} - u_{10} = 0.944 + 0 - 0.518 = 0.426,$$
$$u_{22} = u_{31} + u_{11} - u_{20} = 1.340 + 0.487 - 0.975 = 0.853,$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$u_{92} = u_{10,\,2} + u_{81} - u_{90} = 2.406 + 2.406 - 2.467 = 2.346.$$

Computations on successive layers are carried out analogously.

Problems

In Problems 1 to 3 approximate the solution of the equation

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2},$$

satisfying the conditions

$$u(x, 0) = f(x), \quad u_t(x, 0) = \Phi(x),$$
$$u(0, t) = \varphi(t), \quad u(1, t) = \psi(t)$$

for $0 \leqslant t \leqslant 0.5$, $0 \leqslant x \leqslant 1$, taking the spacing $h = 0.1$ with respect to the argument x.

1. $f(x) = (ax^2 + 1.1) \sin \pi x$, $\Phi(x) = 0$, $\varphi(t) = \psi(t) = 0$, $a = 1.1 + 0.1 \cdot n$, $n = 0, 1, 2, 3, 4$.

To compute the values of u_{i1} make use of the third method (p. 310).

2. $\varphi(t) = \psi(t) = 0$, $\Phi(x) = 0$, the function $f(x)$ is given by the table

x	0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
$f(x)$	0	0.0145	0.0511	$0.0921a$	0.1114	$0.1825/a$	0.1902	$0.1481a$	0.1028	0.0502	0

$a = 0.95 + 0.025 \cdot n$, $n = 1, 2, 3, 4$.

3. $\varphi(t) = 0$, $\psi(t) = f(1)$, the functions $\Phi(x)$, $f(x)$ are given by the table $a = 1.1; 1.2; \ldots; 1.5$.

x	0	0.1	0.2	0.3	0.4	0.5
$f(x)$	0	0.1101	$0.1345 \cdot a$	0.1498	0.1531	$0.1998 \cdot a$
$\Phi(x)$	0	0.0420	0.0500	$0.0510 \cdot a$	0.0440	0.0380
x		0.6	0.7	0.8	0.9	1.0
$f(x)$		0.1402	0.1722	$0.1438 \cdot a$	0.1241	0.1200
$\Phi(x)$		0.0220	$0.0210 \cdot a$	0.0200	0.0190	0

4. Approximate the solution of the equation

$$\frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} = x + t$$

with the spacing $h = l = 0.1$, if $f(x) = (1.5x^2 + 1.2) \sin \pi x$, $\Phi(x) = 0.1x$, $\varphi(t) = \psi(t) = 0$. To compute the values of u_{i1} make use of the first method (p. 309).

5. Approximate the solution of the equation

$$\frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} = x^2 - t^2$$

with the spacing $h = l = 0.1$, if $f(x) = (1.5x^2 + 0.9) e^{-x}$, $\Phi(x) = 0$, $\varphi(t) = 0$, $\psi(t) = 2.4e^{-1}$. To compute the values of u_{i1} take advantage of the first method.

10.8 SOLVING FREDHOLM'S EQUATIONS BY THE METHOD OF FINITE SUMS

Let us consider Fredholm's integral equations of the first kind

$$\int_a^b K(x, s)\, y(s)\, ds = f(x) \tag{1}$$

and of the second kind

$$y(x) - \lambda \int_a^b K(x, s)\, y(s)\, ds = f(x). \tag{2}$$

The method of finite sums (see [2], [13], [17]) consists in replacing the definite integral by a finite sum with the aid of one of the quadrature formulas

$$\int_a^b F(x)\, dx \approx \sum_{j=1}^{n} A_j F(x_j), \tag{3}$$

where x_j are the abscissas of the points belonging to the interval $[a, b]$, A_j $(j = 1, 2, \ldots, n)$ are the coefficients of the quadrature formula independent of $F(x)$. Replacing approximately the integral in Fredholm's equations (1), (2) by formula (3) and putting $x = x_i$, we have, respectively

$$\sum_{j=1}^{n} A_j K_{ij} y_j = f_i \quad (i = 1, 2, \ldots, n), \tag{4}$$

$$y_i - \lambda \sum_{j=1}^{n} A_j K_{ij} y_j = f_i \quad (i = 1, 2, \ldots, n), \tag{5}$$

where $y_i = y(x_i)$, $K_{ij} = K(x_i, x_j)$, $f_i = f(x_i)$.

Hence, we have obtained the systems of linear algebraic equations with respect to the values of y_i. Solving them by one of the known methods (say, the Gaussian method, the method of iterations), we get a table of approximate values of y_i at points x_i. This enables us to write the approximate solution of equation (1) in the form of an interpolation polynomial, and the approximate solution of equation (2) in the form

$$y(x) = f(x) + \lambda \sum_{j=1}^{n} A_j K(x, x_j)\, y_j. \tag{6}$$

Depending on the choice of the quadrature formula (3), we have the following values of coefficients A_j and abscissas x_j:

(1) for *the trapezoidal formula*

$$h = \frac{b-a}{n}, \quad A_0 = A_n = \frac{h}{2}, \quad A_j = h \quad (j = 1, 2, \ldots, n-1),$$
$$x_j = a + jh \quad (j = 0, 1, \ldots, n),$$

(2) for the *Simpson formula*

$$n=2m,\quad h=\frac{b-a}{2m},\quad A_0=A_{2m}=\frac{h}{3},\quad A_1=A_3=\ldots=A_{2m-1}=\frac{4h}{3},$$

$$A_2=A_4=\ldots=A_{2m-2}=\frac{2h}{3},\quad x_j=a+jh\quad (j=0,\ 1,\ \ldots,\ 2m),$$

(3) for the *Gauss formula*

$$A_j=(b-a)\,A_j^{(n)},\quad x_j=a+(b-a)\,x_j^{(n)},$$

where $x_j^{(n)}$ are the Gauss abscissas, and $A_j^{(n)}$ are the Gauss coefficients for the interval (0.1).

The error of the approximate solution depends on the error of the quadrature formula chosen. For the choice of the quadrature formulas see [17]. Error estimate for this method is given in [2], [17].

Example 1. Using the Simpson quadrature formula for $n=2$, approximate the solution of the integral equation

$$y(x)+\int_0^1 xe^{xs}\,y(s)\,ds=e^x. \tag{7}$$

Solution. For the Simpson formula we have

$$h=\frac{1}{2m}=\frac{1}{2},\quad A_0=A_2=\frac{1}{6},\quad A_1=\frac{2}{3},\quad x_0=0,\quad x_1=0.5,\quad x_2=1,$$

hence for equation (7) we may write

$$y(x)+\frac{1}{6}\left[xe^{0\cdot x}y_0+4xe^{0.5\cdot x}y_1+xe^{1\cdot x}y_2\right]=e^x.$$

Putting in the last equality $x=x_i$, we get a system

$$\begin{aligned}
&y_0=1,\\
&y_1+\frac{0.5}{6}\left(y_0+4e^{0.25}y_1+e^{0.5}y_2\right)=e^{0.5},\\
&y_2+\frac{1}{6}\left(y_0+4e^{0.5}y_1+e^1y_2\right)=e^1,
\end{aligned}$$

which after simplifications, takes the form

$$\begin{aligned}
&y_0=1,\\
&1.4280y_1+0.1374y_2=1.5654,\\
&1.0991y_1+1.4530y_2=2.5516.
\end{aligned}$$

Solving this system, we get $y_0=1$, $y_1=1.0002$, $y_2=0.0995$. For the sake of comparison, let us note that the exact solution of equation (7) is the function $y(x)\equiv 1$.

Using formula (6), write the approximate solution of equation (7) in the form

$$y(x)=e^x-\frac{x}{6}\left(1+4.001e^{\frac{x}{2}}+1.000e^x\right).$$

Example 2. Applying the Gauss quadrature formula for $n=2$, approximate the solution of the integral equation

$$y(x)-\frac{1}{2}\int_0^1 e^{xs}\,y(s)\,ds=1-\frac{1}{2x}(e^x-1). \tag{8}$$

Solution. For the Gauss formula we have

$$A_1=A_2=\frac{1}{2}, \quad x_1=0.2113, \quad x_2=0.7887.$$

Replacing equation (8) by system (5) we get

$$y_1-\frac{1}{4}(e^{x_1^2}y_1+e^{x_1x_2}y_2)=1-\frac{1}{2x_1}(e^{x_1}-1),$$
$$y_2-\frac{1}{4}(e^{x_1x_2}y_1+e^{x_2^2}y_2)=1-\frac{1}{2x_2}(e^{x_2}-1).$$

Substituting the values of x_1 and x_2 and performing all necessary transformations, we obtain the following system

$$0.7386y_1-0.2954y_2=0.4434, \quad -0.2954y_1+0.5343y_2=0.2384.$$

Solving this system, we get $y_1=0.9997$, $y_2=0.9990$. For the comparison purposes note that the exact solution of the given equation is the function $y(x)\equiv 1$. Thus, the approximate solution of equation (8) can be written in the form

$$y(x)=0.2499e^{0.2113x}+0.2497e^{0.7887x}+1-\frac{1}{2x}(e^x-1).$$

Example 3. Taking advantage of the rectangles formula for $n=12$, find the approximate solution of the equation

$$y(x)+\int_{-\pi}^{\pi}\frac{y(s)\,ds}{6.8-3.2\cos(x-s)}=25-16\sin^2 x. \tag{9}$$

Solution. For the rectangles formula with $n=12$ we have $h=2\pi/12=\pi/6$, $A_j=\pi/6$. Thus, for equation (9) we get

$$y_i+\frac{\pi}{6}\sum_{j=1}^{n}K_{ij}y_j=25-16\sin^2 x_i. \tag{10}$$

The values of K_{ij} are given in Table 117.

The number of unknowns in the obtained system can be considerably reduced taking into consideration that the required solution is a symmetric one. It can be shown, that if the function $y(x)$ is the solution of equation (9), then the function $y(-x)$ is also the solution of this equation. Therefore, by virtue of uniqueness of the solution of an integral equation we have $y(-x)=y(x)$, i. e.

the solution $y(x)$ is an even function. Denote

$$I(x)=\int_{-\pi}^{\pi}\frac{y(s)\,ds}{6.8-3.2\cos(x+s)}$$

and show that

$$I(x)=I(-x)=I(\pi-x). \tag{11}$$

Indeed

$$I(-x)=\int_{-\pi}^{\pi}\frac{y(s)\,ds}{6.8-3.2\cos(-x+s)}=\int_{-\pi}^{\pi}\frac{y(s)\,ds}{6.8-3.2\cos(x-s)}.$$

Substituting the variables $-s=t$ and taking into account that the function $y(x)$ is even, we have

$$I(-x)=-\int_{\pi}^{-\pi}\frac{y(-t)\,dt}{6.8-3.2\cos(x+t)}=\int_{-\pi}^{\pi}\frac{y(t)\,dt}{6.8-3.2\cos(x+t)}=I(x).$$

Then,

$$I(\pi-x)=\int_{-\pi}^{\pi}\frac{y(s)\,ds}{6.8-3.2\cos(\pi-x+s)}=\int_{-\pi}^{\pi}\frac{y(s)\,ds}{6.8-3.2\cos(x-(\pi+s))},$$

whence, with the aid of the substitution $\pi+s=-t$, we obtain

$$I(\pi-x)=-\int_{\pi}^{-\pi}\frac{y(t)\,dt}{6.8-3.2\cos(x+t)}=\int_{-\pi}^{\pi}\frac{y(t)\,dt}{6.8-3.2\cos(x+t)}=I(x).$$

From (11) it follows

$$y(x)=y(-x)=y(\pi-x),$$

i.e. the graph of the required solution is symmetric with respect to the straight lines $x=0$, $x=\pm\pi/2$ (Fig. 21); by virtue of sym-

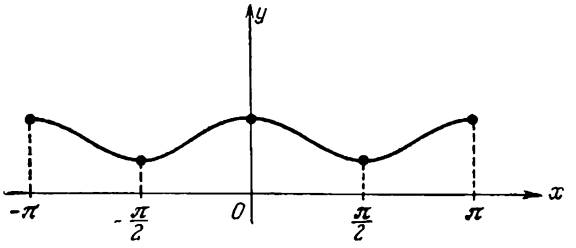

Fig. 21

metry we can write

$$\left.\begin{aligned} &y(-\pi)=y(0)=y(\pi), \quad y\left(-\frac{\pi}{2}\right)=y\left(\frac{\pi}{2}\right), \\ &y\left(-\frac{5}{6}\pi\right)=y\left(-\frac{\pi}{6}\right)=y\left(\frac{\pi}{6}\right)=y\left(\frac{5}{6}\pi\right), \\ &y\left(-\frac{2}{3}\pi\right)=y\left(-\frac{\pi}{3}\right)=y\left(\frac{\pi}{3}\right)=y\left(\frac{2}{3}\pi\right). \end{aligned}\right\} \tag{12}$$

Denoting $y_1=y(0)$, $y_2=y(\pi/6)$, $y_3=y(\pi/3)$, $y_4=y(\pi/2)$ and taking into account conditions (12), we rewrite system (10) as

$$\begin{aligned} &y_1+\frac{\pi}{6}\left[y_1(K(0,-\pi)+K(0,0))+y_2\left(K\left(0,-\frac{5}{6}\pi\right)+\right.\right. \\ &+K\left(0,-\frac{\pi}{6}\right)+K\left(0,\frac{\pi}{6}\right)+K\left(0,\frac{5}{6}\pi\right)\Big)+y_3\left(K\left(0,-\frac{2}{3}\pi\right)+\right. \\ &\left.+K\left(0,-\frac{\pi}{3}\right)+K\left(0,\frac{\pi}{3}\right)+K\left(0,\frac{2}{3}\pi\right)\right)+y_4\left(K\left(0,-\frac{\pi}{2}\right)+\\ &\left.\left.+K\left(0,\frac{\pi}{2}\right)\right)\right]=25, \\ &y_2+\frac{\pi}{6}\left[y_1\left(K\left(\frac{\pi}{6},-\pi\right)+K\left(\frac{\pi}{6},0\right)\right)+y_2\left(K\left(\frac{\pi}{6},-\frac{5}{6}\pi\right)+\right.\right. \\ &+K\left(\frac{\pi}{6},-\frac{\pi}{6}\right)+K\left(\frac{\pi}{6},\frac{\pi}{6}\right)+K\left(\frac{\pi}{6},\frac{5}{6}\pi\right)\Big)+ \\ &+y_3\left(K\left(\frac{\pi}{6},-\frac{2}{3}\pi\right)+K\left(\frac{\pi}{6},-\frac{\pi}{3}\right)+K\left(\frac{\pi}{6},\frac{\pi}{3}\right)+\right. \\ &\left.\left.+K\left(\frac{\pi}{6},\frac{2}{3}\pi\right)\right)+y_4\left(K\left(\frac{\pi}{6},-\frac{\pi}{2}\right)+K\left(\frac{\pi}{6},\frac{\pi}{2}\right)\right)\Big]=21, \\ &y_3+\frac{\pi}{6}\left[y_1\left(K\left(\frac{\pi}{3},-\pi\right)+K\left(\frac{\pi}{3},0\right)\right)+y_2\left(K\left(\frac{\pi}{3},-\frac{5}{6}\pi\right)+\right.\right. \\ &+K\left(\frac{\pi}{3},-\frac{\pi}{6}\right)+K\left(\frac{\pi}{3},\frac{\pi}{6}\right)+K\left(\frac{\pi}{3},\frac{5}{6}\pi\right)\Big)+ \\ &+y_3\left(K\left(\frac{\pi}{3},-\frac{2}{3}\pi\right)+K\left(\frac{\pi}{3},-\frac{\pi}{3}\right)+K\left(\frac{\pi}{3},\frac{\pi}{3}\right)+\right. \\ &\left.\left.+K\left(\frac{\pi}{3},\frac{2}{3}\pi\right)\right)+y_4\left(K\left(\frac{\pi}{3},-\frac{\pi}{2}\right)+K\left(\frac{\pi}{3},\frac{\pi}{2}\right)\right)\Big]=13, \\ &y_4+\frac{\pi}{6}\left[y_1\left(K\left(\frac{\pi}{2},-\pi\right)+K\left(\frac{\pi}{2},0\right)\right)+y_2\left(K\left(\frac{\pi}{2},-\frac{5}{6}\pi\right)+\right.\right. \\ &+K\left(\frac{\pi}{2},-\frac{\pi}{6}\right)+K\left(\frac{\pi}{2},\frac{\pi}{6}\right)+K\left(\frac{\pi}{2},\frac{5}{6}\pi\right)\Big)+ \\ &+y_3\left(K\left(\frac{\pi}{2},-\frac{2}{3}\pi\right)+K\left(\frac{\pi}{2},-\frac{\pi}{3}\right)+K\left(\frac{\pi}{2},\frac{\pi}{3}\right)+\right. \\ &\left.\left.+K\left(\frac{\pi}{2},\frac{2}{3}\pi\right)\right)+y_4\left(K\left(\frac{\pi}{2},-\frac{\pi}{2}\right)+K\left(\frac{\pi}{2},\frac{\pi}{2}\right)\right)\Big]=9, \end{aligned}$$

where the values of $K(x_i, x_j)=K_{ij}$ are given in Table 115.

Table 115

Values of K_{ij} for the Equation (9)

x_i \ x_j	$-\pi$	$-\frac{5}{6}\pi$	$-\frac{2}{3}\pi$	$-\frac{\pi}{2}$	$-\frac{\pi}{3}$	$-\frac{\pi}{6}$	0	$\frac{\pi}{6}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2}{3}\pi$	$\frac{5}{6}\pi$
0	0.100	0.105	0.119	0.147	0.192	0.247	0.278	0.247	0.192	0.147	0.119	0.105
$\frac{\pi}{6}$	0.105	0.119	0.147	0.192	0.247	0.278	0.247	0.192	0.147	0.119	0.105	0.100
$\frac{\pi}{3}$	0.119	0.147	0.192	0.247	0.278	0.247	0.192	0.147	0.119	0.105	0.100	0.105
$\frac{\pi}{2}$	0.147	0.192	0.247	0.278	0.247	0.192	0.147	0.119	0.105	0.100	0.105	0.119

Substituting the values of K_{ij} and computing the coefficients of y_i $(i=1, 2, 3, 4)$, we get the system

$$\begin{aligned}
1.19y_1 + 0.35y_2 + 0.31y_3 + 0.15y_4 &= 25,\\
0.18y_1 + 1.34y_2 + 0.32y_3 + 0.16y_4 &= 21,\\
0.16y_1 + 0.32y_2 + 1.34y_3 + 0.18y_4 &= 13,\\
0.15y_1 + 0.31y_2 + 0.35y_3 + 1.19y_4 &= 9,
\end{aligned}$$

wherefrom we find

$$y_1 = 16.04, \quad y_2 = 12.27, \quad y_3 = 4.73, \quad y_4 = 0.953.$$

Taking advantage of conditions (12), we can get the values of y at the remaining points as well, and the following may be written as the approximate solution of equation (9)

$$y(x) = \frac{\pi}{6} \sum_{j=1}^{12} \frac{y_j}{6.8 - 3.2\cos(x + x_j)}.$$

For the sake of comparison we give the values of the exact solution of

$$\tilde{y}(x) = 8.50 + 7.53\cos 2x$$

at the corresponding points:

$$\tilde{y}(0) = 16.030, \quad \tilde{y}\left(\frac{\pi}{6}\right) = 12.265, \quad \tilde{y}\left(\frac{\pi}{3}\right) = 4.735, \quad \tilde{y}\left(\frac{\pi}{2}\right) = 0.970.$$

Problems

Using the mentioned quadrature formulas, find approximate solutions of the following integral equation.

1. $y(x)+\int_0^1 \frac{y(s)}{1+x^2+s^2}\,ds = 1.5-x^2$ (the trapezoidal formula for $n=4$).

2. $y(x)+\int_0^{0.5} \frac{(1+s)\,y(s)}{2+\sin \pi(x+s)}\,ds = 1+\sin \pi x$ (the trapezoidal formula for $n=5$).

3. $y(x)-\int_0^{0.96} \frac{(1+x+s)\,y(s)}{2+x^2+s^2}\,ds = e^{-x}$ (Simpson's formula for $n=4$).

4. $y(x)-\int_0^1 \frac{1+x+s}{2+xs}\,y(s)\,ds = 1-x^2$ (Gauss' formula for $n=4$).

10.9 SOLVING VOLTERRA'S EQUATION OF THE SECOND KIND BY THE METHOD OF FINITE SUMS

Consider Volterra's integral equation of the second kind

$$y(x)-\lambda\int_a^x K(x, s)\,y(s)\,ds = f(x). \tag{1}$$

It is known (see [30], [38]), that if the kernel $K(x, s)$ is a continuous function in the domain $R\{a\leqslant s\leqslant x\leqslant b\}$, and $f(x)$ is continuous on the interval $[a, b]$, then integral equation (1) has a unique solution for any λ. Choose one of the Newton-Cotes quadrature formulas

$$\int_a^b F(x)\,dx \approx \sum_{j=0}^{n} A_j F(x_j), \tag{2}$$

where x_j are the abscissas of the points belonging to the interval $[a, b]$, and A_j the coefficients of the quadrature formula ($j=0, 1, \ldots, n$). Putting in (1) $x=x_i$, and replacing then approximately the definite integrals by finite sums, we have

$$y_i-\lambda\sum_{j=0}^{i} A_j^{(i)} K_{ij} y_j = f_i \quad (i=0, 1, \ldots, n), \tag{3}$$

where $y_i=y(x_i)$, $K_{ij}=K(x_i, x_j)$, $f_i=f(x_i)$.

Thus, we have obtained a linear system with a triangular matrix. The latter circumstance considerably facilitates its solution. The values of the coefficients A_j depend on the choice of the quadrature formula (see Sec. 10.8).

Example 1. Using the trapezoidal formula with the spacing $h=0.2$ on the interval $[0, 1]$, approximate the solution of the equation

$$y(x)-\int_0^x e^{-x-s}y(s)\,ds=0.5\,(e^{-x}+e^{-3x}). \tag{4}$$

Solution. For the trapezoidal formula at $n=5$ we have

$$A_0=A_5=h/2, \quad A_j=h \quad (j=1, 2, 3, 4).$$

Putting in equation (4) $x=x_i$ $(i=0, 1, \ldots, 5)$, we get

$$y_0=f_0,$$
$$y_i-\int_0^{x_i} e^{-x_i-s}y(s)\,ds=0.5\,(e^{-x_i}+e^{-3x_i}) \quad (i=1, 2, \ldots, 5).$$

Applying the trapezoidal formula with the spacing $h=0.2$ to the definite integrals, we have

$$\left.\begin{aligned}
&y_0=f_0,\\
&y_1-\frac{h}{2}(K_{10}y_0+K_{11}y_1)=f_1,\\
&y_2-\frac{h}{2}(K_{20}y_0+2K_{21}y_1+K_{22}y_2)=f_2,\\
&y_3-\frac{h}{2}[K_{30}y_0+2(K_{31}y_1+K_{32}y_2)+K_{33}y_3]=f_3,\\
&y_4-\frac{h}{2}[K_{40}y_0+2(K_{41}y_1+K_{42}y_2+K_{43}y_3)+K_{44}y_4]=f_4,\\
&y_5-\frac{h}{2}[K_{50}y_0+2(K_{51}y_1+K_{52}y_2+K_{53}y_3+K_{54}y_4)+K_{55}y_5]=f_5.
\end{aligned}\right\} \tag{5}$$

Table 116

Values of K_{ij}, f_i

i	K_{0i}	K_{1i}	K_{2i}	K_{3i}	K_{4i}	K_{5i}	f_i
0	1.00000	0.81873	0.67032	0.54881	0.44933	0.36788	1.00000
1	0.81873	0.67032	0.54881	0.44933	0.36788	0.30119	0.68377
2	0.67032	0.54881	0.44933	0.36788	0.30119	0.24660	0.48576
3	0.54881	0.44933	0.36788	0.30119	0.24660	0.20190	0.35706
4	0.44933	0.36788	0.30119	0.24660	0.20190	0.16530	0.27002
5	0.36788	0.30119	0.24660	0.20190	0.16530	0.13534	0.20883

We then form a table of values of $K(x, s) = e^{-x-s}$, $f(x) = 0.5(e^{-x} + e^{-3x})$ (Table 118) and from system (5) we successively find

$$y_0 = f_0 = 1.0000,$$

$$y_1 = \left[f_1 + \frac{h}{2} K_{10} y_0\right]\left(1 - \frac{h}{2} K_{11}\right)^{-1} = 0.8206,$$

$$y_2 = \left[f_2 + \frac{h}{2} K_{20} y_0 + h K_{21} y_1\right]\left(1 - \frac{h}{2} K_{22}\right)^{-1} = 0.6731,$$

$$y_3 = \left[f_3 + \frac{h}{2} K_{30} y_0 + h(K_{31} y_1 + K_{32} y_2)\right]\left(1 - \frac{h}{2} K_{33}\right)^{-1} = 0.5518,$$

$$y_4 = \left[f_4 + \frac{h}{2} K_{40} y_0 + h(K_{41} y_1 + K_{42} y_2 + K_{43} y_3)\right]\left(1 - \frac{h}{2} K_{44}\right)^{-1} = 0.4522,$$

$$y_5 = \left[f_5 + \frac{h}{2} K_{50} y_0 + h(K_{51} y_1 + K_{52} y_2 + K_{53} y_3 + K_{54} y_4)\right] \times \left(1 - \frac{h}{2} K_{55}\right)^{-1} = 0.3705.$$

Problems

Applying the trapezoidal formula with $h = 0.2$, find the approximate solutions of the following integral equations.

1. $y(x) - 0.5 \int_0^x \frac{y(s)}{2 + \sin \pi (x + s)}\, ds = 2 - \sin \pi x$ on the interval [0, 1].

2. $y(x) - \int_0^x \frac{y(s)}{(1 + x + y)}\, ds = (1 + x)$ on the interval [0, 1].

3. $y(x) - \int_0^x \frac{y(s)}{1 + e^{-xs}}\, ds = \cosh x$ on the interval [0; 1.2].

10.10 THE METHOD OF REPLACING THE KERNEL BY A DEGENERATE ONE

Consider Fredholm's integral equation of the second kind

$$y(x) - \lambda \int_a^b K(x, s)\, y(s)\, ds = f(x). \tag{1}$$

The kernel $K(x, s)$ is called *degenerate* if it is representable in the form

$$K(x, s) = \sum_{i=1}^{n} \alpha_i(x)\, \beta_i(s), \tag{2}$$

where the functions $\alpha_i(x)$ and $\beta_i(s)$ $(i = 1, 2, \ldots, n)$ are linearly independent on the interval $[a, b]$.

The suggested method (see [2], [13], [17]) is based on that the integral equation (10.1) with a degenerate kernel can be supplied with the exact solution. Replace approximately the kernel $K(x, s)$ by a degenerate one

$$K(x, s) \approx \sum_{i=1}^{n} \alpha_i(x)\beta_i(s) \tag{3}$$

and look for an approximate solution of equation (1) in the form

$$y(x) = f(x) + \lambda \sum_{i=1}^{n} c_i \alpha_i(x), \tag{4}$$

where

$$c_i = \int_a^b \beta_i(s)\, y(s)\, ds. \tag{5}$$

Substituting expression (4) into (5), we get

$$c_i = \int_a^b \beta_i(s) f(s)\, ds + \lambda \int_a^b \beta_i(s) \sum_{j=1}^{n} c_j \alpha_j(s)\, ds \quad (i = 1, 2, \ldots, n).$$

Introducing the notation

$$f_i = \int_a^b \beta_i(s) f(s)\, ds, \quad A_{ij} = \int_a^b \alpha_j(s)\beta_i(s)\, ds, \tag{6}$$

we have

$$c_i - \lambda \sum_{j=1}^{n} c_j A_{ij} = f_i \quad (i = 1, 2, \ldots, n). \tag{7}$$

We get a system of linear algebraic equations with respect to c_i. Solving this system, write the approximate solution of equation (1) in the form of (4). We can take an interval of Taylor's series or a Fourier series for the function $K(x, s)$ as a degenerate kernel. For more detail see [2] and [17].

Example 1. Find an approximate solution on the equation

$$y(x) - \int_0^1 \sinh(xs)\, y(s)\, ds = 1 - x^2. \tag{8}$$

Solution. Replace the kernel $K(x, s) = \sinh(x, s)$ by the sum of the first three terms of Taylor's series

$$\sinh(xs) \approx xs + \frac{(xs)^3}{3!} + \frac{(xs)^5}{5!}.$$

Then we shall seek for a solution of equation (8) in the form

$$y(x)=1-x^2+c_1x+c_2x^3+c_3x^5.$$

Putting $f(x)=1-x^2$, $\alpha_1=x$, $\alpha_2=x^3$, $\alpha_3=x^5$, $\beta_1(s)=s$, $\beta_2(s)=\frac{s^3}{3!}$, $\beta_3(s)=\frac{s^5}{5!}$, we find the coefficients of system (7) by formulas (6)

$$f_1=\int_0^1\beta_1(s)\,f(s)\,ds=\int_0^1(s-s^5)\,ds=\frac{1}{4},$$

$$f_2=\int_0^1\beta_2(s)\,f(s)\,ds=\int_0^1\frac{1}{3!}(s^3-s^5)\,ds=\frac{1}{72},$$

$$f_3=\int_0^1\beta_3(s)\,f(s)\,ds=\int_0^1\frac{1}{5!}(s^5-s^7)\,ds=\frac{1}{2880},$$

$$A_{11}=\int_0^1 s^2\,ds=\frac{1}{3},\quad A_{12}=\int_0^1 s^4\,ds=\frac{1}{5},\quad A_{13}=\int_0^1 s^4\,ds=\frac{1}{7},$$

$$A_{21}=\int_0^1\frac{1}{3!}s^4\,ds=\frac{1}{30},\quad A_{22}=\int_0^1\frac{1}{3!}s^6\,ds=\frac{1}{42},\quad A_{23}=\int_0^1\frac{s^8}{3!}ds=\frac{1}{54},$$

$$A_{31}=\int_0^1\frac{1}{5!}s^6\,ds=\frac{1}{840},\quad A_{32}=\int_0^1\frac{s^8}{5!}ds=\frac{1}{1080},\quad A_{33}=\int_0^1\frac{s^{10}}{5!}ds=\frac{1}{1320}.$$

Thus, we have the following system

$$c_1=\frac{1}{3}c_1+\frac{1}{5}c_2+\frac{1}{7}c_3+\frac{1}{4},$$

$$c_2=\frac{1}{30}c_1+\frac{1}{42}c_2+\frac{1}{54}c_3+\frac{1}{72},$$

$$c_3=\frac{1}{840}c_1+\frac{1}{1080}c_2+\frac{1}{1320}c_3+\frac{1}{2880}.$$

Solving it by the method of iteration, we get $c_1=0.3833$, $c_2=0.0273$, $c_3=0.0008$. Hence, the approximate solution of equation (8) can be written in the form

$$y(x)=1-x^2+0.3833x+0.0273x^3+0.0008x^5.$$

Problems

Find approximate solutions of the integral equations, replacing the kernel by the sum of the first three terms of Taylor's series.

1. $y(x)-\int_0^1\frac{\sin(axs)}{s}y(s)\,ds=f(x)$, $a=0.6+0.2\cdot k$, $k=0, 1, 2, 3$.

(a) $f(x)=x$, (b) $f(x)=\frac{1}{2\sqrt{x}}$.

2. $y(x)-\int_0^1 (1+s)(e^{axs}-1)\,y(s)\,ds=f(x)$, $a_1=0.2+0.1\cdot k$, $k=0, 1, 2$, $a_2=-0.4+0.1\cdot n$, $n=0, 1, 2$.

(a) $f(x)=\frac{1}{x}$, (b) $f(x)=(1-x)$, (c) $f(x)=e^{-x}$.

3. $y(x)-\int_0^1 \frac{xs}{\sqrt{1+axs}}\,y(s)\,ds=f(x)$, $a_1=0.1$, $a_2=0.2$, $a_3=-0.1$, $a_4=-0.2$.

(a) $f(x)=1+x$, (b) $f(x)=e^{-x}$, (c) $f(x)=\sqrt{x}$.

APPENDICES

1. FREQUENTLY USED CONSTANTS

Quantity	n	log n	Quantity	n	log n
π	3.1415 93	0.49715	$1:\pi$	0.3183 10	$\bar{1}$.50285
2π	6.2831 85	0.79818	$1:2\pi$	0.1591 55	$\bar{1}$.20182
3π	9.4247 78	0.97427	$1:3\pi$	0.1061 03	$\bar{1}$.02573
4π	12.5663 71	1.09921	$1:4\pi$	0.0795 77	$\bar{2}$.90079
$\pi:2$	1.5707 96	0.19612	$2:\pi$	0.6366 20	$\bar{1}$.80388
$\pi:3$	1.0471 98	0.02003	$3:\pi$	0.9549 30	$\bar{1}$.97997
$\pi:4$	0.7853 98	$\bar{1}$.89509	$4:\pi$	1.2732 40	0.10491
$\pi:6$	0.5235 99	$\bar{1}$.71900	$6:\pi$	1.9098 59	0.28100
$\pi:180\,(=1°)$	0.0174 53	$\bar{2}$.24188	$180°:\pi$	57°,2957 80	1.75812
$\pi:10800\,(=1')$	0.0002 91	$\bar{4}$.46373	$10\,800':\pi$	3437',74 68	3.53627
$\pi:648\,000\,(=1'')$	0.0000 05	$\bar{6}$.68557	$648\,000'':\pi$	206264'', 81	5.31443
π^2	9.8696 04	0.99430	$1:\pi^2$	0.1013 21	$\bar{1}$.00570
$\sqrt{\pi}$	1.7724 54	0.24857	$\sqrt{1:\pi}$	0.5641 90	$\bar{1}$.75143
$\sqrt{2\pi}$	2.5066 28	0.39909	$\sqrt{1:2\pi}$	0.3989 42	$\bar{1}$.60091
$\sqrt{\pi:2}$	1.2533 14	0.09806	$\sqrt{2:\pi}$	0.7978 85	$\bar{1}$.90194
$\sqrt[3]{\pi}$	1.4615 92	0.16572	$\sqrt[3]{1:\pi}$	0.6827 84	$\bar{1}$.83428
$\sqrt{4\pi:3}$	1.6119 92	0 20736	$\sqrt[3]{3:4\pi}$	0.6203 50	$\bar{1}$.79264
e	2.7182 82	0.43429	$1:e$	0.3678 79	$\bar{1}$.56571
e^2	7.3890 56	0.86859	$1:e^2$	0.1353 35	$\bar{1}$.13141
$\sqrt{e}$	1.6487 21	0.21715	$\sqrt{1:e}$	0.6065 31	$\bar{1}$.78285
$\sqrt[3]{e}$	1.3956 12	0.14476	$\sqrt[3]{1:e}$	0.7165 32	$\bar{1}$.85524
$e^{\pi:2}$	4.8104 77	0.68219	$e^{-\pi:2}$	0.2078 80	$\bar{1}$.31781
e^{π}	23.1406 93	1.36438	$e^{-\pi}$	0.0432 14	$\bar{2}$.63562
$e^{2\pi}$	535.4916 56	2.72875	$e^{-2\pi}$	0.0018 67	$\bar{3}$.27125
C *	0.5772 16	$\bar{1}$.76134	$\ln\pi$	1.1447 30	0.05870
$M=\log e$	0.4342 94	$\bar{1}$.63778	$1:M=\ln 10$	2.3025 85	0.36222
g **	9.81	0.99167	$1:g$	0.10194	$\bar{1}$.00833
g^2	96.2361	1.98334	$1:2g$	0.050968	$\bar{2}$.70730
$\sqrt{g}$	3.13209	0.49583	$\pi\sqrt{g}$	9.83976	0.99298
$\sqrt{2g}$	4.42945	0.64635	$\pi\sqrt{2g}$	13.91552	1.14350

* C — Euler's constant.

** g — acceleration due to gravity in m/sec²; the value of g is given here in round figures for sea level at a latitude of 45°–50°.

2. *Table of Random Numbers*

10 09 73 25 33	76 52 01 35 86	34 67 35 48 76	80 95 90 91 17	39 29 27 49 45
37 54 20 48 05	64 89 47 42 96	24 80 52 40 37	20 63 61 04 02	00 82 29 16 65
08 42 26 89 53	19 64 50 93 03	23 20 90 25 60	15 95 33 47 64	35 08 03 36 06
99 01 90 25 29	09 37 67 07 15	38 31 13 11 65	88 67 67 43 97	04 43 62 76 59
12 80 79 99 70	80 15 73 61 47	64 03 23 66 53	98 95 11 68 77	12 17 17 68 33
66 06 57 47 17	34 07 27 68 50	36 69 73 61 70	65 81 33 98 85	11 19 92 91 70
31 06 01 08 05	45 57 18 24 06	35 30 34 26 14	86 79 90 74 39	23 40 30 97 32
85 26 97 76 02	02 05 16 56 92	68 66 57 48 18	73 05 38 52 47	18 62 38 85 79
63 57 33 21 35	05 32 54 70 48	90 55 35 75 48	28 46 82 87 09	83 49 12 56 24
73 79 64 57 53	03 52 96 47 78	35 80 83 42 82	60 93 52 03 44	35 27 38 84 35
98 52 01 77 67	14 90 56 86 07	22 10 94 05 58	60 97 09 34 33	50 50 07 39 98
11 80 50 54 31	39 80 82 77 32	50 72 56 82 48	29 40 52 42 01	52 77 56 78 51
83 45 29 96 34	06 28 89 80 83	13 74 67 00 78	18 47 54 06 10	68 71 17 78 17
88 68 54 02 00	86 50 75 84 01	36 76 66 79 51	90 36 47 64 93	29 60 91 10 62
99 59 46 73 48	87 51 76 49 69	91 82 60 89 28	93 78 56 13 68	23 47 83 41 13
65 48 11 76 74	17 46 85 09 50	58 04 77 69 74	73 03 95 71 86	40 21 81 65 44
80 12 43 56 35	17 72 70 80 15	45 31 82 23 74	21 11 57 82 53	14 38 55 37 63
74 35 09 98 17	77 40 27 72 14	43 23 60 02 10	45 52 16 42 37	96 28 60 26 55
69 91 62 68 03	66 25 22 91 48	36 93 68 72 03	76 62 11 39 90	94 40 05 64 18
09 89 32 05 05	14 22 56 85 14	46 42 75 67 88	96 29 77 88 22	54 38 21 45 98
91 49 91 45 23	68 47 92 76 86	46 16 28 35 54	94 75 08 99 23	37 08 92 00 48
80 33 69 45 98	26 94 03 68 58	70 29 73 41 35	53 14 03 33 40	42 05 08 23 41
44 10 48 19 49	85 15 74 79 54	32 97 92 65 75	57 60 04 08 81	22 22 20 64 13
12 55 07 37 42	11 10 00 20 40	12 86 07 46 97	96 64 48 94 39	28 70 72 58 15
63 60 64 93 29	16 50 53 44 84	40 21 95 25 63	43 65 17 70 82	07 20 73 17 90
61 19 69 04 46	26 45 74 77 74	51 92 43 37 29	65 39 45 95 93	42 58 26 05 27
15 47 44 52 66	95 27 07 99 53	59 36 78 38 48	82 39 61 01 18	33 21 15 94 66
94 55 72 85 73	67 89 75 43 87	54 62 24 44 31	91 19 04 25 92	92 92 74 59 73
42 48 11 62 13	97 34 40 87 21	16 86 84 87 67	03 07 11 20 59	25 70 14 66 70
23 52 37 83 17	73 20 88 98 37	68 93 59 14 16	26 25 22 96 63	05 52 28 25 62
04 49 35 24 94	75 24 63 38 24	45 86 25 10 25	61 96 27 93 35	65 33 71 24 72
00 54 99 76 54	64 05 18 81 59	96 11 96 38 96	54 69 28 23 91	23 28 72 95 29
35 96 31 53 07	26 89 80 93 54	33 35 13 54 62	77 97 45 00 24	90 10 33 93 33
59 80 80 83 91	45 42 72 68 42	83 60 94 97 00	13 02 12 48 92	78 56 32 01 06
46 05 88 52 36	01 39 09 22 86	77 28 14 40 77	93 91 08 36 47	70 61 74 29 41
32 17 90 05 97	87 37 92 52 41	05 56 70 70 07	86 74 31 71 57	85 39 41 18 38
69 23 46 14 06	20 11 74 52 04	15 95 66 00 00	18 74 39 24 23	97 11 89 63 38
19 56 54 14 30	01 75 87 53 79	40 41 92 15 85	66 67 43 68 06	84 96 28 52 07
45 15 51 49 38	19 47 60 72 46	43 66 79 45 43	59 04 79 00 33	20 82 66 95 41
94 86 43 19 94	36 16 81 08 51	34 88 88 15 53	01 54 03 54 56	05 01 45 11 76
98 08 62 48 26	45 24 02 84 04	44 99 90 88 96	39 09 47 34 07	35 44 13 18 80
33 18 51 62 32	41 94 15 09 49	89 43 54 85 81	88 69 54 19 94	37 54 87 30 43
80 95 10 04 06	96 38 27 07 74	20 15 12 33 87	25 01 62 52 98	94 62 46 11 71
79 75 24 91 40	71 96 12 82 96	69 86 10 25 91	74 85 22 05 39	00 38 75 95 79
18 63 33 25 37	98 14 50 65 71	31 01 02 46 74	05 45 56 14 27	77 93 89 19 36
74 02 94 39 02	77 55 73 22 70	97 79 01 71 19	52 52 75 80 21	80 81 45 17 48
54 17 84 56 11	80 99 33 71 43	05 33 51 29 69	56 12 71 92 55	36 04 09 03 24
11 66 44 98 83	52 07 98 48 27	59 38 17 15 39	09 97 33 34 40	88 46 12 33 56
48 32 47 79 28	31 24 96 47 10	02 29 53 68 70	32 30 75 75 46	15 02 00 99 94
69 07 49 41 38	87 63 79 19 76	35 58 40 44 01	10 51 82 16 15	01 84 87 69 38
09 18 82 00 97	32 82 53 95 27	04 22 08 63 04	83 38 98 73 74	64 27 85 80 44
90 04 58 54 97	51 98 15 06 54	94 93 88 19 97	91 87 07 61 50	68 47 66 46 59
73 18 95 02 07	47 67 72 62 69	62 29 06 44 64	27 12 46 70 18	41 36 18 27 60
75 76 87 64 90	20 97 18 17 49	90 42 91 22 72	95 37 50 58 71	93 82 34 31 78
54 01 64 40 56	66 28 13 10 03	00 68 22 73 98	20 71 45 32 95	07 70 61 78 13

Table of Random Numbers (Concluded)

08 35 86 99 10	78 54 24 27 85	13 66 15 88 73	04 61 89 75 53	31 22 30 84 20
28 30 60 32 64	81 33 31 05 91	10 51 00 78 93	32 60 16 01 75	94 11 90 18 40
53 84 08 62 33	81 59 41 36 28	51 21 59 02 90	28 46 66 87 95	77 76 22 07 91
91 75 75 37 41	61 61 36 22 69	50 26 39 02 12	55 78 17 65 14	83 48 34 70 55
89 41 59 26 94	00 39 75 83 91	12 60 71 76 46	48 94 97 23 06	94 54 13 74 08
77 51 30 38 20	86 83 42 99 01	68 41 48 27 74	51 90 81 39 80	72 89 35 55 07
19 50 23 71 74	69 97 92 02 88	55 21 02 97 73	74 28 77 52 51	65 34 46 74 15
21 81 85 93 13	93 27 88 17 57	05 68 67 31 56	07 08 28 50 46	31 85 33 84 52
51 47 46 64 99	68 10 72 36 21	94 04 99 13 45	42 83 60 91 91	08 00 74 54 49
99 55 96 83 31	62 53 52 41 70	69 77 71 28 30	74 81 97 81 42	43 86 07 28 34
33 71 34 80 07	93 58 47 28 69	51 92 66 47 21	58 30 32 98 22	93 17 49 39 72
85 27 48 68 93	11 30 32 92 70	28 83 43 41 37	73 51 59 04 00	71 14 84 36 43
84 13 38 96 40	44 03 55 21 66	73 85 27 00 91	61 22 26 05 61	62 32 71 84 23
56 73 21 62 34	17 39 59 61 31	10 12 39 16 22	85 49 65 75 60	81 60 41 88 80
65 13 85 68 06	87 64 88 52 61	34 31 36 58 61	45 87 52 10 69	85 64 44 72 77
38 00 10 21 76	81 71 91 17 11	71 60 29 29 37	74 21 96 40 49	65 58 41 96 98
37 40 29 63 97	01 30 47 75 86	56 27 11 00 86	47 32 46 26 05	40 03 03 74 38
97 12 54 03 48	87 08 33 14 17	21 81 53 92 50	75 23 76 20 47	15 50 12 95 78
21 82 64 11 34	47 14 33 40 72	64 63 88 59 02	49 13 90 64 41	03 85 65 45 52
73 13 54 27 42	95 71 90 90 35	85 79 47 42 96	08 78 98 81 56	64 69 11 92 02
07 63 87 79 29	03 06 11 80 72	96 20 74 41 56	23 82 19 95 38	04 71 30 69 94
60 52 88 34 41	07 95 41 98 14	59 17 52 06 95	05 53 35 21 39	61 21 20 64 55
83 59 63 56 55	06 95 89 29 83	05 12 80 97 19	77 43 35 37 83	92 30 15 04 98
10 85 06 27 46	99 59 91 05 07	13 19 90 63 19	53 07 57 18 39	06 41 01 93 62
39 82 09 89 52	43 62 26 31 47	64 42 18 08 14	43 80 00 93 51	31 02 47 31 67
59 58 00 64 78	75 56 97 88 00	88 83 55 41 86	23 76 80 61 56	04 11 10 84 08
38 50 80 73 41	23 79 34 87 63	90 82 29 70 22	17 71 90 42 07	95 95 44 99 53
30 69 27 06 68	91 68 81 61 27	36 19 68 00 91	82 06 76 34 00	05 46 26 92 00
65 44 39 56 59	18 28 82 74 37	49 63 22 40 41	08 33 76 56 76	96 29 99 08 36
27 26 76 02 64	13 19 27 22 94	07 47 74 46 06	17 98 54 89 11	97 34 13 03 58
91 30 70 69 91	19 07 22 42 10	36 69 95 37 28	78 82 53 57 93	28 97 66 62 52
68 43 49 46 88	84 47 31 36 22	62 12 69 81 08	12 84 38 25 90	09 81 59 31 46
48 90 81 58 77	54 74 52 45 91	35 70 00 47 51	83 82 45 26 92	54 13 05 51 60
06 91 34 51 97	42 67 27 86 01	11 88 30 95 28	63 01 19 89 01	14 97 44 03 41
10 45 51 60 19	14 21 03 37 12	91 34 23 78 21	88 32 58 08 51	43 66 77 08 83
12 88 39 73 43	65 02 76 11 84	04 28 50 13 92	17 97 41 50 77	90 71 22 67 69
21 77 83 09 76	38 30 73 69 61	31 61 94 20 96	63 28 10 20 23	08 81 64 74 49
19 52 35 95 15	65 12 25 96 59	86 28 36 82 58	69 57 21 37 98	16 43 59 15 29
67 24 55 26 70	35 58 31 65 63	79 21 68 66 86	76 46 33 42 22	26 65 59 08 02
60 58 44 73 77	07 50 03 79 92	15 13 42 65 29	26 76 08 36 37	41 32 64 43 44
53 85 31 13 77	36 06 69 48 50	58 83 87 38 59	49 36 47 33 31	96 24 04 36 42
24 63 73 87 36	74 38 18 93 12	52 62 30 79 92	12 36 91 86 01	03 74 28 38 73
83 08 01 21 51	38 99 22 28 15	07 75 95 17 77	97 37 72 75 85	51 97 23 78 67
16 44 42 43 31	36 15 19 90 73	27 49 37 09 39	85 13 03 25 52	54 84 65 47 59
60 79 01 81 57	57 17 86 57 62	11 16 17 85 76	45 81 95 29 79	65 13 00 48 60

ANSWERS

CHAPTER 1

1.1

1. (a) $a=2.15$, $\Delta=0.14\cdot10^{-2}$, $\delta=0.65\cdot10^{-3}$, (b) $a=0.162$, $\Delta=0.48\cdot10^{-3}$, $\delta=0.3\cdot10^{-2}$, (c) $a=0.0120$, $\Delta=0.4\cdot10^{-4}$, $\delta=0.33\cdot10^{-2}$, (d) $a=1.23$, $\Delta=0.5\cdot10^{-2}$, $\delta=0.41\cdot10^{-2}$, (e) $a=-0.00153$, $\Delta=0.19\cdot10^{-5}$, $\delta=0.12\cdot10^{-2}$, (f) $a=-393$, $\Delta=0.15$, $\delta=0.38\cdot10^{-3}$, (g) $a=0.155$, $\Delta=0.5\cdot10^{-3}$, $\delta=0.32\cdot10^{-2}$, (h) $a=0.00392$, $\Delta=0.2\cdot10^{-5}$, $\delta=0.51\cdot10^{-3}$, (i) $a=626$, $\Delta=0.45$, $\delta=0.72\cdot10^{-3}$, (j) $a=94.5$, $\Delta=0.25\cdot10^{-1}$, $\delta=0.26\cdot10^{-3}$.

2. (a) $\Delta=0.13\cdot10^{2}$, (b) $\Delta=0.16\cdot10^{-1}$, (c) $\Delta=0.36$, (d) $\Delta=0.9\cdot10^{-1}$, (e) $\Delta=0.23\cdot10$.

3. $\delta_{\alpha_1}=0.13\cdot10^{-4}$, $\delta_{\alpha_2}=0.62\cdot10^{-5}$, $\delta_{\alpha_3}=0.28\cdot10^{-3}$, $\delta_{\alpha_4}=0.37\cdot10^{-5}$.

4. (a) 2, (b) 3, (c) 4, (d) 3, (e) 2, (f) 5, (g) 1, (h) 2, (i) 4, (j) 2.

5. (a) 3, (b) 2, (c) 1, (d) 2, (e) 1, (f) 3, (g) 1, (h) 2, (i) 1, (j) 2.

1.2

1. (a) 399, $\Delta=0.9$, $\delta=0.23\%$, (b) 205.0, $\Delta=0.07$, $\delta=0.034\%$, (c) 326.6, $\Delta=0.09$, $\delta=0.028\%$, (d) 19.5, $\Delta=0.6$, $\delta=3\%$.

1.3

1. (a) 30, $\delta=0.72\%$, $\Delta=0.2$, (b) 43.7, $\delta=0.23\%$, $\Delta=0.1$, (c) 0.3, $\delta=28\%$, $\Delta=0.1$, (d) $138\cdot10^{2}$, $\delta=0.34\%$, $\Delta=0.48\cdot10^{2}$, (e) 19, $\delta=0.87\%$, $\Delta=0.22$, (f) $18\cdot10^{3}$, $\delta=0.96\%$, $\Delta=0.18\cdot10^{3}$.

2. (a) 1.130, $\delta=0.019\%$, $\Delta=0.2\cdot10^{-3}$, (b) 0.12, $\delta=4.6\%$, $\Delta=0.55\cdot10^{-2}$, (c) 54, $\delta=13\%$, $\Delta=7$, (d) 0.877, $\delta=0.06\%$, $\Delta=0.53\cdot10^{-3}$, (e) 20.7, $\delta=0.14\%$, $\Delta=0.29\cdot10^{-1}$, (f) 0.0016, $\delta=0.7\%$, $\Delta=0.11\cdot10^{-4}$.

3. 19.9 m^2 with the absolute error 0.1 m^2.

4. 0.4800 with the absolute error 0.00044.

5. $\Delta=12\pi\ cm^2=38\ cm^2$, $\delta=8.3\%$.

6 $\Delta=3.84\ cm^3$, $\delta=0.75\%$. **7.** $\delta=1.5\%$.

1.4

1. $y = \sin x$, (a) $\Delta_y = 0.28 \cdot 10^{-3}$, $\delta_y = 0.13 \cdot 10^{-2}$, $\sin x = 0.196$, (b) $\Delta_y = 0.16 \cdot 10^{-4}$, $\delta_y = 0.21 \cdot 10^{-4}$, $\sin x = 0.7513$, (c) $\Delta_y = 0.2 \cdot 10^{-3}$, $\delta_y = 0.29 \cdot 10^{-3}$, $\sin x = 0.707$, (d) $\Delta_y = 0.56 \cdot 10^{-3}$, $\delta_y = 0.72 \cdot 10^{-3}$, $\sin x = 0.768$, (e) $\Delta_y = 0.45 \cdot 10^{-2}$, $\delta_y = 0.9 \cdot 10^{-2}$, $\sin x = 0.43$, (f) $\Delta_y = 0.44 \cdot 10^{-4}$, $\delta_y = 0.22 \cdot 10^{-4}$, $\sin x = 0.8979$.

$y = \cos x$. (a) $\Delta_y = 0.57 \cdot 10^{-4}$, $\delta_y = 0.11 \cdot 10^{-4}$, $\cos x = 0.9805$, (b) $\Delta_y = 0.18 \cdot 10^{-4}$, $\delta_y = 0.2 \cdot 10^{-4}$, $\cos x = 0.6600$, (c) $\Delta_y = 0.2 \cdot 10^{-3}$, $\delta_y = 0.29 \cdot 10^{-3}$, $\cos x = 0.707$, (d) $\Delta_y = 0.67 \cdot 10^{-3}$, $\delta_y = 0.8 \cdot 10^{-3}$, $\cos x = 0.64$, (e) $\Delta_y = 0.21 \cdot 10^{-2}$, $\delta_y = 0.1 \cdot 10^{-2}$, $\cos x = 0.90$, (f) $\Delta_y = 0.9 \cdot 10^{-4}$, $\delta_y = 0.18 \cdot 10^{-3}$, $\cos x = 0.4402$.

$y = \tan x$. (a) $\Delta_y = 0.3 \cdot 10^{-4}$, $\delta_y = 0.15 \cdot 10^{-2}$, $\tan x = 0.2004$, (b) $\Delta_y = 0.55 \cdot 10^{-4}$, $\delta_y = 0.48 \cdot 10^{-4}$, $\tan x = 1.1383$, (c) $\Delta_y = 0.58 \cdot 10^{-3}$, $\delta_y = 0.12 \cdot 10^{-2}$, $\tan x = 1.0$, (d) $\Delta_y = 0.21 \cdot 10^{-2}$, $\delta_y = 0.43 \cdot 10^{-2}$, $\tan x = 1.20$, (e) $\Delta_y = 0.62 \cdot 10^{-2}$, $\delta_y = 0.16 \cdot 10^{-1}$, $\tan x = 0.48$, (f) $\Delta_y = 0.5 \cdot 10^{-3}$, $\delta_y = 0.12 \cdot 10^{-2}$, $\tan x = 2.040$.

2. (a) $y = 8.379$, $\Delta_y = 0.19 \cdot 10^{-2}$, $\delta_y = 0.23 \cdot 10^{-3}$, (b) $y = 3.598$, $\Delta_y = 0.11 \cdot 10^{-2}$, $\delta_y = 0.31 \cdot 10^{-3}$, (c) $y = 0.1046$, $\Delta_y = 0.37 \cdot 10^{-3}$, $\delta_y = 0.36 \cdot 10^{-3}$.

3. (a) $u = 0.81$, $\Delta_u = 0.27 \cdot 10^{-2}$, $\delta_u = 0.33 \cdot 10^{-2}$, (b) $u = 3.66$, $\Delta_u = 0.1 \cdot 10^{-2}$, $\delta_u = 0.27 \cdot 10^{-2}$, (c) $u = 7.48$, $\Delta_u = 0.49 \cdot 10^{-2}$, $\delta_u = 0.64 \cdot 10^{-3}$.

4. $\delta = 0.12 \cdot 10^{-2} = 0.12\%$.

1.5

1. (a) 20″, (b) 20″, (c) 4′, (d) $0.2 \cdot 10^{-1}$, (e) $0.1 \cdot 10^{-2}$.

2. (a) $0.50 \cdot 10^{-5}$, (b) $0.52 \cdot 10^{-5}$, (c) $0.68 \cdot 10^{-5}$, (d) $0.78 \cdot 10^{-5}$, (e) $0.12 \cdot 10^{-4}$, (f) $0.96 \cdot 10^{-4}$.

3. (a) $0.5 \cdot 10^{-2}$, (b) $0.5 \cdot 10^{-3}$, (c) $0.2 \cdot 10^{-4}$, (d) $0.4 \cdot 10^{-4}$, (e) $0.7 \cdot 10^{-1}$.

4. (a) 6, (b) 6, (c) 5.

5. 4.

6. (a) $\Delta_{x_1} = 0.56 \cdot 10^{-4}$, $\Delta_{x_2} = 0.25 \cdot 10^{-1}$, (b) $\Delta_{x_1} = 0.28 \cdot 10^{-4}$, $\Delta_{x_2} = 0.15 \cdot 10^{-4}$, $\Delta_{x_3} = 0.77 \cdot 10^{-5}$, (c) $\Delta_{x_1} = 0.90 \cdot 10^{-5}$, $\Delta_{x_2} = 0.85 \cdot 10^{-5}$, $\Delta_{x_3} = 0.62 \cdot 10^{-5}$.

7. $\delta_l = 0.2\%$, $\delta_s = 0.7\%$.

CHAPTER 2

2.1

2.

ξ	$P(\xi)$	ξ	$P(\xi)$
0.85	— 2.1178	1.35	—14.0310
0.90	— 2.7669	1.40	—16.0938
0.95	— 3.4984	1.45	—18.3621
1.00	— 4.3100	1.50	—20.8487
1.05	— 5.2958	1.55	—23.5666
1.10	— 6.3761	1.60	—26.5288
1.15	— 7.5917	1.65	—29.7489
1.20	— 8.9536	1.70	—33.2404
1.25	—10.4729	1.75	—37.0172
1.30	—12.1615	1.80	—41.0934
		1.85	—45.4832

2.2

1.

(e)

x	e^{x^2}
0.50	1.28402
0.52	1.31049
0.54	1.33857
0.56	1.36834
0.58	1.39990
0.60	1.43333
0.62	1.46873
0.64	1.50622
0.66	1.54589
0.68	1.58788
0.70	1.63232
0.72	1.67934
0.74	1.72910
0.76	1.78176
0.78	1.83749
0.80	1.89648

(f)

x	e^{-x^2}
1.30	0.18452
1.31	0.17977
1.32	0.17510
1.33	0.17052
1.34	0.16603
1.35	0.16162
1.36	0.15730
1.37	0.15306
1.38	0.14891
1.39	0.14484
1.40	0.14086
1.41	0.13696
1.42	0.13313
1.43	0.12939
1.44	0.12573
1.45	0.12215

(g)

x	$\frac{1}{2\sqrt{\pi}} e^{-x^2/2}$
1.78	0.0579
1.81	0.0548
1.84	0.0519
1.87	0.0491
1.90	0.0464
1.93	0.0438
1.96	0.0413
1.99	0.0389
2.02	0.0367
2.05	0.0345
2.08	0.0324
2.11	0.0304
2.14	0.0286
2.17	0.0268
2.20	0.0251
2.23	0.0235

x	$\frac{1}{2\sqrt{\pi}} e^{-x^2/2}$
2.545	0.01106
2.550	0.01092
2.555	0.01079
2.560	0.01065
2.565	0.01051
2.570	0.01038
2.575	0.01025
2.580	0.01012
2.585	0.00998
2.590	0.00986
2.595	0.00973
2.600	0.00960
2.605	0.00948
2.610	0.00936
2.615	0.00924
2.620	0.00912

2.

(e)

x	$\frac{\sin x}{x}$	x	$\frac{\sin x}{x}$	x	$\frac{\sin x}{x}$
0.40	0.97355	0.46	0.96510	0.51	0.95721
0.41	0.97222	0.47	0.96359	0.52	0.95554
0.42	0.97086	0.48	0.96204	0.53	0.95384
0.43	0.96947	0.49	0.96046	0.54	0.95210
0.44	0.96804	0.50	0.95885	0.55	0.95034
0.45	0.96659				

(f)

x	$\frac{\cos x}{x}$	x	$\frac{\cos x}{x}$	x	$\frac{\cos x}{x}$
0.25	3.87565	0.31	3.07204	0.36	2.59971
0.26	3.71688	0.32	2.96636	0.37	2.51980
0.27	3.56952	0.33	2.86680	0.38	2.44385
0.28	3.43234	0.34	2.77281	0.39	2.37156
0.29	3.30429	0.35	2.68392	0.40	2.30265
0.30	3.18446				

2.3

1.

(c)

x	$e^{1/x}$
2.0	1.64872
2.5	1.49182
3.0	1.39561
3.5	1.33071
4.0	1.28402
4.5	1.24885
5.0	1.22140
5.5	1.19940
6.0	1.18136
6.5	1.16631
7.0	1.15356
7.5	1.14263
8.0	1.13315
8.5	1.12485
9.0	1.11752
9.5	1.11100

(d)

x	$\frac{1}{2\sqrt{\pi}} e^{-x^2/2}$
0.40	0.26041
0.42	0.25828
0.44	0.25607
0.46	0.25377
0.48	0.25140
0.50	0.24895
0.52	0.24642
0.54	0.24382
0.56	0.24116
0.58	0.23842
0.60	0.23562
0.62	0.23277
0.64	0.22985
0.66	0.22688
0.68	0.22386
0.70	0.22080

x	$\frac{1}{2\sqrt{\pi}} e^{-x^2/2}$
1.2	0.13731
1.3	0.12118
1.4	0.10587
1.5	0.09158
1.6	0.07843
1.7	0.06650
1.8	0.05583
1.9	0.04640
2.0	0.03818
2.1	0.03110
2.2	0.02508
2.3	0.02003
2.4	0.01583
2.5	0.01239
2.6	0.00960
2.7	0.00737

2.5

1.

(c)

x	$1/x^3$	x	$1/x^3$	x	$1/x^3$
3	0.037037	15	0.000296	25	0.000064
5	0.008000	17	0.000204	27	0.000051
7	0.002915	19	0.000146	29	0.000041
9	0.001372	21	0.000108	31	0.000034
11	0.000751	23	0.000082	33	0.000028
13	0.000455				

(d)

x	$x/(1+x)$	x	$x/(1+x)$	x	$x/(1+x)$
0.007	0.006951	0.025	0.024390	0.040	0.038462
0.010	0.009901	0.028	0.027237	0.043	0.041227
0.013	0.012833	0.031	0.030068	0.046	0.043977
0.016	0.015748	0.034	0.032882	0.049	0.046711
0.019	0.018646	0.037	0.035680	0.052	0.049430
0.022	0.021526				

2.

	(a)	(b)
x	$\sqrt{x}$	$x\sqrt{x}$
2	1.414214	2.828427
3	1.732051	5.196152
4	2.000000	8.000000
5	2.236068	11.180340
6	2.449490	14.696938
7	2.645751	18.520259
8	2.828427	22.627417
9	3.000000	27.000000
10	3.162278	31.622777
11	3.316625	36.482873
12	3.464102	41.569219
13	3.605551	46.872167
14	3.741657	52.383203
15	3.872983	58.094750
16	4.000000	64.000000
17	4.123106	70.092796

	(c)	(d)
x	$\sqrt{1+x^2}$	$\sqrt{x^2+1}\,x$
0.300	1.044031	3.480102
0.302	1.044607	3.458964
0.304	1.045187	3.438115
0.306	1.045771	3.417551
0.308	1.046357	3.397265
0.310	1.046948	3.377251
0.312	1.047542	3.357506
0.314	1.048139	3.338023
0.316	1.048740	3.318798
0.318	1.049345	3.299826
0.320	1.049952	3.281101
0.322	1.050564	3.262620
0.324	1.051178	3.244378
0.326	1.051797	3.226370
0.328	1.052418	3.208592
0.330	1.053043	3.191040

3.

(a)

x	$1/\sqrt{x}$
3	0.577350
5	0.447214
7	0.377964
9	0.333333
11	0.301511
13	0.277350
15	0.258199
17	0.242536
19	0.229416
21	0.218218
23	0.208514
25	0.200000
27	0.192450
29	0.185695
31	0.179605
33	0.174078

(b)

x	$1/\sqrt{2+x^2}$
0.300	0.691714
0.302	0.691515
0.304	0.691315
0.306	0.691114
0.308	0.690911
0.310	0.690707
0.312	0.690502
0.314	0.690296
0.316	0.690089
0.318	0.689881
0.320	0.689672
0.322	0.689461
0.324	0.689249
0.326	0.689037
0.328	0.688823
0.330	0.688608

(c)

x	$(2x+1)/\sqrt{x}$
3.100	6.767962
3.105	6.777504
3.110	6.787048
3.115	6.796593
3.120	6.806139
3.125	6.815685
3.130	6.825233
3.135	6.834782
3.140	6.844333
3.145	6.853884
3.150	6.863436
3.155	6.872990
3.160	6.882544
3.165	6.882099
3.170	6.901656
3.175	6.911214

(d)

x	$1/\sqrt{x(x+1)}$
2.300	0.362977
2.302	0.362709
2.304	0.362442
2.306	0.362175
2.308	0.361909
2.310	0.361643
2.312	0.361377
2.314	0.361112
2.316	0.360847
2.318	0.360583
2.320	0.360319
2.322	0.360055
2.324	0.359792
2.326	0.359529
2.328	0.359267
2.330	0.359005

4.

(a)

x	$\sqrt[3]{x}$	x	$\sqrt[3]{x}$	x	$\sqrt[3]{x}$
3	1.442250	9	2.080084	14	2.410142
4	1.587401	10	2.154435	15	2.466212
5	1.709976	11	2.223980	16	2.519842
6	1.817121	12	2.289428	17	2.571282
7	1.912931	13	2.351335	18	2.620741
8	2.000000				

(b)

x	$1/\sqrt[3]{x}$	x	$1/\sqrt[3]{x}$	x	$1/\sqrt[3]{x}$
3	0.693361	9	0.480750	14	0.414913
4	0.629961	10	0.464159	15	0.405480
5	0.584804	11	0.449644	16	0.396850
6	0.550321	12	0.436790	17	0.388911
7	0.522758	13	0.425290	18	0.381571
8	0.500000				

5.

(a)

x	$\sqrt[4]{x}$
0.05	0.472871
0.07	0.514369
0.09	0.547723
0.11	0.575901
0.13	0.600462
0.15	0.622333
0.17	0.642114
0.19	0.660220
0.21	0.676947
0.23	0.692519
0.25	0.707107
0.27	0.720843
0.29	0.733837
0.31	0.746175
0.33	0.757929
0.35	0.769161

(b)

x	$\sqrt[5]{x}$
0.05	0.549280
0.07	0.587516
0.09	0.617801
0.11	0.643100
0.13	0.664950
0.15	0.684255
0.17	0.701600
0.19	0.717382
0.21	0.731887
0.23	0.745325
0.25	0.757858
0.27	0.769514
0.29	0.780692
0.31	0.791175
0.33	0.801130
0.35	0.810613

(c)

x	$\sqrt[6]{x}$
0.05	0.606962
0.07	0.641972
0.09	0.669433
0.11	0.692201
0.13	0.711744
0.15	0.728923
0.17	0.744289
0.19	0.758215
0.21	0.770968
0.23	0.782747
0.25	0.793701
0.27	0.803947
0.29	0.813579
0.31	0.822672
0.33	0.831290
0.35	0.839482

(d)

x	$\sqrt[7]{x}$
0.05	0.651836
0.07	0.683934
0.09	0.708934
0.11	0.729552
0.13	0.747172
0.15	0.762603
0.17	0.776362
0.19	0.788796
0.21	0.800155
0.23	0.810622
0.25	0.820335
0.27	0.829404
0.29	0.837915
0.31	0.845936
0.33	0.853525
0.35	0.860730

CHAPTER 3

3.3

1. $x_1 = 0.5$, $x_2 = 1.3$, $x_3 = 2.5$ (indicated the exact values of the unknowns). **2.** $x_1 = 0.36450$, $x_2 = 0.53053$, $x_3 = -0.40649$. **3.** $x_1 = 1.34$, $x_2 = -4.76$, $x_3 = 2.58$. **4.** $x_1 = 2.06$, $x_2 = 3.22$. $x_3 = 4.03$, $x_4 = -2.01$, $x_5 = 3.00$.

5.

α	β	x_1	x_2	x_3	α	β	x_1	x_2	x_3
0.0	0.0	0.58000	1.45834	0.49908	1.0	0.6	0.61493	1.59977	0.40154
	0.2	0.55593	1.47017	0.49683		0.8	0.58970	1.61248	0.39619
	0.4	0.53185	1.48200	0.49458		1.0	0.56448	1.62519	0.39083
	0.6	0.50577	1.49383	0.49233					
	0.8	0.48369	1.50566	0.49008	1.5	0.0	0.74903	1.62679	0.38373
	1.0	0.45961	1.51749	0.48783		0.2	0.72303	1.64036	0.37685
						0.4	0.69704	1.65374	0.36997
0.5	0.0	0.63462	1.50566	0.45563		0.6	0.67105	1.66713	0.36308
	0.2	0.61003	1.51785	0.45181		0.8	0.64506	1.68052	0.35620
	0.4	0.58544	1.53004	0.44799		1.0	0.61906	1.69391	0.34932
	0.6	0.56086	1.54224	0.44417					
	0.8	0.53627	1.55443	0.44035	2.0	0.0	0.81079	1.70266	0.35306
	1.0	0.51168	1.56663	0.43653		0.2	0.78407	1.71692	0.34461
						0.4	0.75718	1.73118	0.33616
1.0	0.0	0.69060	1.56164	0.41760		0.6	0.73029	1.74544	0.32771
	0.2	0.66538	1.57435	0.41224		0.8	0.70339	1.75970	0.31926
	0.4	0.64015	1.58706	0.40689		1.0	0.67650	1.77395	0.31081

6.

α	β	x_1	x_2	x_3	x_4
0.0	0.0	−3.01480	−1.17641	4.59989	−0.73923
	0.2	−2.94057	−1.04992	4.44130	−0.79040
	0.4	−2.86635	−0.92344	4.28270	−0.84157
	0.6	−2.79212	−0.79695	4.12410	−0.89274
	0.8	−2.71789	−0.67047	3.96550	−0.94391
	1.0	−2.64366	−0.54398	3.80691	−0.99508
0.2	0.0	−2.89809	−0.68363	4.18049	−0.95161
	0.2	−2.82754	−0.56313	4.02394	−0.99553
	0.4	−2.75700	−0.44264	3.86739	−1.03946
	0.6	−2.68645	−0.32215	3.71084	−1.08338
	0.8	−2.61591	−0.20166	3.55429	−1.12731
	1.0	−2.54536	−0.08116	3.39774	−1.17123
0.4	0.0	−2.80040	−0.18579	3.72950	−1.12449
	0.2	−2.73345	−0.07220	3.57619	−1.16140
	0.4	−2.66650	0.04139	3.42287	−1.19831
	0.6	−2.59955	0.15498	3.26956	−1.23522
	0.8	−2.53260	0.26857	3.11624	−1.27214
	1.0	−2.46565	0.38215	2.96293	−1.30905
0.6	0.0	−2.72373	0.30586	3.25743	−1.25443
	0.2	−2.66020	0.41180	3.10845	−1.28472
	0.4	−2.59668	0.51774	2.95947	−1.31501
	0.6	−2.53316	0.62368	2.81049	−1.34530
	0.8	−2.46963	0.72961	2.66151	−1.37559
	1.0	−2.40611	0.83555	2.51253	−1.40588
0.8	0.0	−2.66904	0.78068	2.77541	−1.34003
	0.2	−2.60871	0.87841	2.63173	−1.36422
	0.4	−2.54837	0.97615	2.48805	−1.38840
	0.6	−2.48804	1.07389	2.34438	−1.41259
	0.8	−2.42770	1.17162	2.20070	−1.43678
	1.0	−2.36737	1.26936	2.05702	−1.46096

3.4

2. (a) $x=-1.4000$, $y=0.7406$, (b) $x=-1.590$, $y=0.7409$. 3. $x_1=3.000$, $x_2=-2.000$, $x_3=6.000$. 4. $x_1=1.34$, $x_2=-4.76$, $x_3=2.58$. 5. $x_1=0.3732$, $x_2=0.1415$, $x_3=0.4622$. 6. $x_1=3.04$, $x_2=1.80$, $x_3=2.92$, $x_4=-1.53$, $x_5=3.32$. 7. $x_1=1.7781$, $x_2=-4.3351$, $x_3=2.5304$.

8.

α	β	x_1	x_2	x_3	α	β	x_1	x_2	x_3
0.00	0.00	−1.0890	1.3127	1.4013	0.50	1.05	−1.0178	1.2754	1.3750
	0.35	−0.98110	1.2941	1.3913		1.40	−0.90595	1.2565	1.3647
	0.70	−0.87319	1.2756	1.3813		1.75	−0.79410	1.2377	1.3545
	1.05	−0.76529	1.2570	1.3713					
	1.40	−0.65738	1.2385	1.3613	0.75	0.00	−1.4887	1.3416	1.4082
	1.75	−0.54948	1.2199	1.3514		0.35	−1.3748	1.3227	1.3978
						0.70	−1.2610	1.3137	1.3874
0.25	0.00	−1.2201	1.3222	1.4034		1.05	−1.1471	1.2847	1.3771
	0.35	−1.1103	1.3035	1.3933		1.40	−1.0332	1.2658	1.3667
	0.70	−1.0001	1.2848	1.3832		1.75	−0.91938	1.2468	1.3563
	1.05	−0.89054	1.2661	1.3730					
	1.40	−0.78068	1.2474	1.3629	1.00	0.00	−1.6262	1.3516	1.4108
	1.75	−0.67082	1.2287	1.3528		0.35	−1.5103	1.3325	1.4003
						0.70	−1.3944	1.3134	1.3899
0.50	0.00	−1.3533	1.3318	1.4057		1.05	−1.2785	1.2943	1.3794
	0.35	−1.2415	1.3130	1.3954		1.40	−1.1626	1.2752	1.3689
	0.70	−1.1296	1.2942	1.3852		1.75	−1.0467	1.2560	1.3584

9.

α	β	x_1	x_2	x_3	x_4
0.0	0.0	0.41093	−0.43009	1.1546	1.5463
	0.5	0.19749	−0.41385	1.2676	2.5847
	1.0	−0.04594	−0.39762	1.3805	3.6231
	1.5	−0.28938	−0.38138	1.4935	4.6615
	2.0	−0.53281	−0.36514	1.6065	5.6999
	2.5	−0.77625	−0.34891	1.7195	6.7384
0.5	0.0	0.90593	−0.45260	1.0238	−0.35119
	0.5	1.0809	−0.50835	1.1190	−0.97083
	1.0	1.2559	−0.56410	1.2142	−1.5905
	1.5	1.4309	−0.61985	1.3094	−2.2101
	2.0	1.6058	−0.67559	1.4046	−2.8298
	2.5	1.7808	−0.7314	1.4998	−3.4494
1.0	0.0	0.78167	−0.37945	0.92704	0.080047
	0.5	0.86137	−0.41615	1.0135	−0.15149
	1.0	0.94107	−0.45285	1.10000	−0.38302
	1.5	1.0208	−0.48955	1.1865	−0.61455
	2.0	1.1005	−0.52625	1.2730	−0.84609
	2.5	1.1802	−0.56295	1.3594	−1.0776
1.5	0.0	0.73902	−0.32853	0.84798	0.17381
	0.5	0.79675	−0.35925	0.92673	0.03466
	1.0	0.85448	−0.38997	1.0055	−0.10449
	1.5	0.91221	−0.42070	1.0843	−0.24364
	2.0	0.96994	−0.45142	1.1630	−0.38279
	2.5	1.0277	−0.48214	1.2418	−0.52193

α	β	x_1	x_2	x_3	x_4
2.0	0.0	0.70966	−0.28660	0.78251	0.20954
	0.5	0.75769	−0.31368	0.85478	0.11182
	1.0	0.80571	−0.34076	0.92706	0.01411
	1.5	0.85374	−0.36784	0.99933	−0.08361
	2.0	0.90176	−0.39492	1.0716	−0.18132
	2.5	0.94979	−0.42200	1.1439	−0.27904

3.5

1. $x_1=1.24$, $x_2=-2.45$, $x_3=-2.50$. **2.** $x_1=2.34$, $x_2=4.51$, $x_3=-6.00$, $x_4=-1.30$. **3.** $x_1=1$, $x_2=2$, $x_3=1$, $x_4=-1$, $x_5=4$.

3.6

1. $x_1=1.3$, $x_2=2.2$, $x_3=3.5$ (indicated the exact values of the unknowns). **2.** $x_1=0.74481$, $x_2=0.79206$, $x_3=0.82455$. **3.** $x_1=3.7$, $x_2=-1.5$, $x_3=2.1$, $x_4=1.3$ (indicated the exact values of the unknowns). **4.** $x_1=0.29270$, $x_2=1.4196$, $x_3=1.1799$, $x_4=0.89104$.

5.

α	β	x_1	x_2	x_3	x_4	x_5
0.00	0.00	0.79838	0.22453	0.17223	0.92064	0.51090
	0.35	0.87179	0.23413	0.08283	0.90004	0.59660
	0.70	0.94520	0.24373	−0.00658	0.87944	0.68229
	1.05	1.01860	0.25333	−0.09598	0.85884	0.76798
	1.40	1.09201	0.26293	−0.18538	0.83823	0.85367
	1.75	1.16542	0.27253	−0.27479	0.81763	0.93936
0.25	0.00	0.74312	0.25963	0.15411	1.00146	0.47677
	0.35	0.81306	0.26913	0.07001	0.98056	0.55763
	0.70	0.88299	0.27864	−0.01409	0.95965	0.63848
	1.05	0.95292	0.28814	−0.09820	0.93875	0.71934
	1.40	1.02285	0.29765	−0.18230	0.91784	0.80020
	1.75	1.09279	0.30715	−0.26640	0.89694	0.88105
0.50	0.00	0.69015	0.29830	0.13395	1.09142	0.44328
	0.35	0.75694	0.30781	0.05457	1.07007	0.51986
	0.70	0.82374	0.31733	−0.02481	1.04871	0.59643
	1.05	0.89053	0.32685	−0.10420	1.02736	0.67300
	1.40	0.95732	0.33636	−0.18358	1.00601	0.74957
	1.75	1.02412	0.34588	0.26296	0.98466	0.82614
0.75	0.00	0.63861	0.34149	0.11151	1.19270	0.40994
	0.35	0.70256	0.35113	0.03635	1.17074	0.48269
	0.70	0.76651	0.36077	−0.03881	1.14878	0.55544
	1.05	0.83046	0.37041	−0.11396	1.12681	0.62820
	1.40	0.89440	0.38005	−0.18912	1.10485	0.70095
	1.75	0.95835	0.38969	−0.26427	1.08289	0.77370

α	β	x_1	x_2	x_3	x_4	x_5
1.00	0.00	0.58764	0.39048	0.08637	1.30815	0.37619
	0.35	0.69900	0.40036	0.01501	1.28539	0.44552
	0.70	0.71036	0.41025	−0.05634	1.26263	0.51485
	1.05	0.77172	0.42013	−0.12769	1.23987	0.58418
	1.40	0.83308	0.43002	−0.19904	1.21711	0.65352
	1.75	0.89445	0.43990	−0.27039	1.19435	0.72285

3.7

1.

α	Δ
0.0	−646.416
0.5	−633.677
1.0	−614.329
1.5	−589.120
2.0	−558.802
2.5	−524.123
3.0	−485.835
3.5	−444.686
4.0	−401.428
4.5	−356.809
5.0	−311.581

2.

α	Δ
0.0	−38.5112
0.5	83.4051
1.0	279.202
1.5	566.879
2.0	965.936
2.5	1497.37
3.0	2183.69
3.5	3048.89
4.0	4118.47
4.5	5419.43
5.0	6980.27

3.

α	Δ
0.00	1950.02
0.25	1984.06
0.50	1985.08
0.75	1951.61
1.00	1883.13
1.25	1780.17
1.50	1644.49
1.75	1479.12
2.00	1288.53
2.25	1078.71
2.50	857.342

3.8

1. $\begin{pmatrix} -0.1558 & 0.1448 & -0.2148 & -0.0233 \\ -0.0243 & 0.6957 & -0.6562 & -0.0242 \\ -0.0439 & -0.6183 & -0.7486 & -0.0146 \\ -0.0259 & -0.2596 & 0.2294 & 0.1666 \end{pmatrix}$.

2. $\begin{pmatrix} 0.1327 & 0.0162 & -0.0472 & -0.0228 \\ -0.0511 & 0.1730 & 0.0056 & -0.0348 \\ 0 & -0.0715 & 0.0729 & 0 \\ -0.0414 & -0.0341 & 0.0022 & 0.1796 \end{pmatrix}$.

3. $\begin{pmatrix} 0.93794 & -0.06844 & -0.07961 & -0.08592 \\ -0.08852 & 0.90599 & -0.09919 & -0.10560 \\ -0.11135 & -0.11697 & -0.87842 & -0.12707 \\ -0.13546 & -0.14018 & -0.14381 & 0.85161 \end{pmatrix}$.

4.

α	a_{i1}	a_{i2}	a_{i3}	a_{i4}
0.0	1.20551	−0.03637	0.13541	−0.45050
	−0.24219	0.03422	0.16237	−0.00175
	−0.02266	0.25422	−0.04151	−0.02824
	−4.67126	−0.26049	−0.67114	2.33731

α	a_{i1}	a_{i2}	a_{i3}	a_{i4}
0.5	−0.68679 0.09707 0.00195 2.87191	−0.10254 0.03631 0.22568 0.07418	−0.11822 0.19025 −0.03125 0.35429	0.45249 −0.14780 −0.03532 −1.31347
1.0	−0.24812 0.02273 −0.00284 1.11046	−0.07228 0.02466 0.20199 0.00610	−0.05634 0.16511 −0.02676 0.11865	0.23168 −0.09806 −0.02902 −0.46917
1.5	−0.14538 0.00719 −0.00352 0.69182	−0.05907 0.01808 0.18289 −0.00604	−0.04027 0.15028 −0.02292 0.06453	0.17453 −0.07953 −0.02538 −0.27226
2.0	−0.10010 0.00133 −0.00356 0.50385	−0.05017 0.01359 0.16711 −0.00951	−0.03225 0.13861 −0.01986 0.04132	0.14622 −0.06775 −0.02256 −0.18592

5.

α	a_{i1}	a_{i2}	a_{i3}	α	a_{i1}	a_{i2}	a_{i3}
0.0	0.042334 −0.002881 −0.004658	−0.002399 0.032180 −0.005143	−0.005548 −0.001317 0.035716	0.6	0.041271 −0.002813 −0.004447	−0.002356 0.032172 −0.005041	−0.005297 −0.001298 0.034952
0.2	0.041973 −0.002858 −0.004586	−0.002384 0.032178 −0.005109	−0.005462 −0.001311 0.035457	0.8	0.040928 −0.002791 −0.004380	−0.002342 0.032170 −0.005008	−0.005217 −0.001290 0.034704
0.4	0.041619 −0.002835 −0.004516	−0.002370 0.032175 −0.005075	−0.005379 −0.001305 0.035203				

3.9

3. $x_1 = 1.9031$, $x_2 = 3.2007$, $x_3 = 5.0428$, $x_4 = 0.1432$. 4. $x_1 = 2.9274$, $x_2 = 3.0505$, $x_3 = 4.8286$.

5.

α	β	x_1	x_2	x_3	α	β	x_1	x_2	x_3
0.0	0.0	0.9444	1.1743	1.1775	0.4	0.6	0.9327	1.1567	1.1657
	0.2	0.9449	1.1679	1.1786		0.8	0.9331	1.1502	1.1667
	0.4	0.9454	1.1614	1.1796		1.0	0.9336	1.1438	1.1677
	0.6	0.9459	1.1550	1.1806					
	0.8	0.9463	1.1485	1.1816	0.6	0.0	0.9248	1.1768	1.1554
	1.0	0.9468	1.1421	1.1827		0.2	0.9253	1.1704	1.1564
						0.4	0.9257	1.1639	1.1574
0.2	0.0	0.9378	1.1751	1.1700		0.6	0.9262	1.1575	1.1584
	0.2	0.9383	1.1687	1.1711		0.8	0.9267	1.1511	1.1594
	0.4	0.9387	1.1623	1.1721		1.0	0.9271	1.1446	1.1604
	0.6	0.9392	1.1558	1.1731					
	0.8	0.9397	1.1494	1.1741	0.8	0.0	0.9184	1.1776	1.1481
	1.0	0.9402	1.1430	1.1752		0.2	0.9189	1.1712	1.1491
						0.4	0.9194	1.1648	1.1501
0.4	0.0	0.9312	1.1760	1.1626		0.6	0.9198	1.1583	1.1512
	0.2	0.9317	1.1695	1.1637		0.8	0.9203	1.1519	1.1522
	0.4	0.9322	1.1631	1.1647		1.0	0.9208	1.1454	1.1532

CHAPTER 4

4.1

1.

α	k				
	0.0	0.1	0.2	0.3	0.4
0.5	0.66293 0.62460	0.79656 0.58427	0.91099 0.54085	1.0145 0.49265	1.1077 0.43962
0.6	0.64621 0.61215	0.77208 0.56672	0.87646 0.51918	0.96799 0.46786	1.0484 0.41262
0.7	0.63103 0.60053	0.75057 0.55030	0.84723 0.49877	0.93000 0.44417	1.0013 0.38617
0.8	0.61711 0.58963	0.73135 0.53484	0.82180 0.47943	0.89775 0.42145	0.96195 0.36036
0.9	0.60428 0.57938	0.71396 0.52021	0.79926 0.46102	0.86964 0.39960	0.92814 0.33519

2.

α	k				
	0.6	0.7	0.8	0.9	1.0
1.0	0.27241 0.05822	0.33256 0.08271	0.38791 0.10778	0.43933 0.13289	0.48748 0.15772
1.1	0.26301 0.13345	0.32249 0.15332	0.37727 0.17435	0.42824 0.19588	0.47606 0.21752
1.2	0.24614 0.20804	0.30636 0.22311	0.36161 0.24007	0.41291 0.25806	0.46100 0.27655
1.3	0.22065 0.28401	0.28346 0.29357	0.34043 0.30607	0.39298 0.32030	0.44208 0.33552
1.4	0.18348 0.36450	0.25202 0.36669	0.31258 0.37377	0.36764 0.38367	0.41856 0.39527

4.2

1. $x=3.487$, $y=2.262$. **2.** $x=1.0000$, $y=2.0000$.

3.

α	1.0	1.5	2.0	2.5	3.0	3.5
x y	1.5020 1.5456	1.3388 1.6124	1.2343 1.6615	1.1590 1.7006	1.1010 1.7333	1.0544 1.7613

4. $x_1=-2.0000$, $y_1=2.0000$, $x_2=1.3508$, $y_2=0.59214$. **5.** $x=1.7913$, $y=-0.34422$.

CHAPTER 5

5.2

1. $|R_1|<10^{-4}$. (a) 1.6603, (b) 1.6686, (c) 1.6871, (d) 1.7074, (e) 1.7177, (f) 1.7437, (g) 1.7648, (h) 1.7701, (i) 1.8022, (j) 1.8112.

2. $|R_2|<0.16\cdot10^{-4}$, (a) 0.91317, (b) 0.93841, (c) 0.97462, (d) 0.98880, (e) 0.99946, (f) 0.99927, (g) 0.98703, (h) 0.96155, (i) 0.93653, (j) 0.89596, (k) 0.82409, (l) 0.79777, (m) 0.73698, (n) 0.64772, (o) 0.61595.

3. $|R_1|<0.36\cdot10^{-4}$. **4.** $|R_1|\leqslant0.125\cdot10^{-6}$.

5. $|R_1|<0.7\cdot10^{-7}$.

6. (1) (a) 0.50674, (b) 0.50725, (c) 0.51043, (d) 0.51077, (e) 0.45818, (f) 0.45783, (g) 0.45761, (h) 0.45696;

(2) (a) 0.5732, (b) 0.5807, (c) 0.5880, (d) 0.5956, (e) 1.5313, (f) 1.5433, (g) 1.5510, (h) 1.5642;

(3) (a) 0.29504, (b) 0.29261, (c) 0.29780, (d) 0.29907, (e) 0.60127, (f) 0.60[illegible], (g) 0.61001, (h) 0.61190.

5.3

1. (a) 0.48358, (b) 0.48267, (c) 0.48245, (d) 0.48161, (e) 0.48104, (f) 0.48055, (g) 0.48019, (h) 0.47973, (i) 0.47941, (j) 0.47861, (k) 0.48876, (l) 0.48811, (m) 0.48747, (n) 0.48695, (o) 0.48612, (p) 0.48595, (q) 0.48579, (r) 0.48535, (s) 0.48482, (t) 0.48420.
2. (a) 0.99116, (b) 0.99988, (c) 1.00825, (d) 1.01464, (e) 1.02762, (f) 1.04631, (g) 1.05596, (h) 1.06596, (i) 1.07614, (j) 0.89488, (k) 0.89795, (l) 0.90588, (m) 0.91574, (n) 0.92577, (o) 0.94097, (p) 0.94444, (q) 0.95460, (r) 0.96622, (s) 0.97389.
3. (a) 0.44812, (b) 0.45038, (c) 0.45551, (d) 0.45906, (e) 0.46109, (f) 0.46237, (g) 0.46388, (h) 0.47189, (i) 0.47370, (j) 0.47707, (k) 0.41959, (l) 0.41893, (m) 0.42293, (n) 0.42433, (o) 0.42967, (p) 0.43175, (q) 0.43610, (r) 0.43886, (s) 0.44206.

(a)

x	$f(x)$
0.85	0.8274
0.95	0.7868
1.05	0.7428
1.15	0.6957

(b)

x	$f(x)$
1.35	1.5102
1.45	1.5989
1.55	1.6971
1.65	1.8056

(c)

x	$f(x)$
2.35	2.1958
2.45	2.4050
2.55	2.6334
2.65	2.8829

5.4

1. (a) $-14.2x^2+28.67x+91.37$, (b) x^3+x^2-x+2, (c) $-0.0205x^3-0.02x^2+2.73x+1.45$, (d) $-4.1x^5+337.8x^4-11\,283.9x^3+182\,940.4x^2-1\,460\,817x+4\,593\,561.7$, (e) $x^2-10x+1$, (f) x^4-3x^2+2.
2. (a) 3.899, (b) 3.937, (c) 3.975, (d) 4.012, (e) 4.087.
3. (a) 6.055, (b) 6.223, (c) 6.527, (d) 6.938, (e) 7.211, (f) 7.294.
4. $\sin\frac{\pi}{12}=0.2588\pm0.3\cdot10^{-3}$. 5. $\cos\frac{\pi}{5}=0.9397\pm0.3\cdot10^{-4}$.
6. (a) 49.46, (b) 15.38, (c) 778687.
7. (a) 1.39316, (b) 1.40175, (c) 1.40865, (d) 1.42252, (e) 1.44876, (f) 1.46466, (g) 1.47711, (h) 1.50404, (i) 1.48247, (j) 1.49502, (k) 1.50043, (l) 1.46999.

5.5

1. (a) 2.122, (b) 5.2.
2. (a) 1.021, (b) 1.045, (c) 1.063, (d) 1.072, (e) 1.113, (f) 1.128, (g) 1.162, (h) 1.185, (i) 1.215, (j) 1.239.
3. (a) 0.525, (b) 0.541, (c) 0.568, (d) 0.587, (e) 0.623, (f) 0.631, (g) 0.668, (h) 0.679.
4. (a) 1.075, (b) 1.123, (c) 1.138, (d) 1.159, (e) 1.175, (f) 1.218, (g) 1.221, (h) 1.248, (i) 1.269.
5. (a) 0.055, (b) 0.082, (c) 0.123.
6. (a) 1.113, (b) 1.138, (c) 1.167, (d) 1.195, (e) 1.221, (f) 1.242.
7. (a) 0.082, (b) 0.091, (c) 0.113.

5.6

1. 0.6529. 2. 0.6507. 3. 1.8411. 4. 0.21331. 5. 1.47767. 6. 0.9286. 7. 0.2427. 8. 0.8767. 9. 0.7391. 10. 0.4384. 11. 0.22106. 12. 0.7208. 13. 0.87550. 14. 0.37696. 15. 0.7872.

16.

Variant No.	1	2	3	4	5	6	7	8	9	10
x	1.010	1.053	1.024	0.978	0.929	0.474	0.642	0.676	0.675	0.662

17.

Variant No.	1	2	3	4	5	6	7	8	9	10
x	0.6180	0.4943	0.4162	0.3683	0.5143	0.4196	0.3584	0.3201	0.2814	0.2556

18.

Variant No.	1	2	3	4	5	6	7	8	9	10
x	0.8894	0.8300	0.7688	0.7051	0.6378	0.6063	0.5796	0.5533	0.4983	0.4701

19. 1.7320. **20.** −0.7321. **21.** 0.6180. **22.** 0.2679. **23.** −3.1623. **24.** 1.2361. **25.** −2.3028. **26.** 3.6457. **27.** 1.6180.

CHAPTER 6

6.1

1. (a)

x	y'	x	y'	x	
1.2	0.714	1.8	1.317	2.4	2.298
1.3	0.797	1.9	1.448	2.5	2.517
1.4	0.886	2.0	1.591	2.6	2.756
1.5	0.982	2.1	1.745	2.7	3.016
1.6	1.085	2.2	1.914	2.8	3.301
1.7	1.197	2.3	2.098		

Note. Computed by formula (7).

(b)

x	y''	x	x	y''
1.2	0.81	1.8	1.25	2.4	2.08
1.3	0.85	1.9	1.38	2.5	2.30
1.4	0.92	2.0	1.48	2.6	2.49
1.5	1.00	2.1	1.61	2.7	2.72
1.7	1.07	2.2	1.77	2.8	2.98
1.6	1.16	2.3	1.92	2.9	3.27

Note. Computed by formula (12).

(c) 0.566; 0.636; 3.612; 3.954. (d) 0.67; 3.56.

2. (a)

x	y'	x	y'	x	y'
0.8	6.276	1.8	10.017	2.8	15.772
1.0	6.904	2.0	11.975	3.0	17.261
1.2	7.587	2.2	12.021	3.2	18.889
1.4	8.329	2.4	13.163	3.4	20.671
1.6	9.137	2.6	14.410		

Note. Computed by formula (7).

(b)

x	y''	x	y''	x	y''
0.8	3.021	1.8	4.589	2.8	7.120
1.0	3.272	2.0	5.001	3.0	7.782
1.2	3.558	2.2	5.461	3.2	8.514
1.4	3.870	2.4	5.964	3.4	9.313
1.6	4.212	2.6	6.513		

Note. Computed by formula (9), taking into account differences up to the fourth order included.

(c) 5.160; 5.695; 22.620; 24.755. (d) 2.570; 2.800; 10.181; 11.161.

6.2

1. (a) The truncation error may be neglected, the rounding off error being equal to $\frac{3 \cdot 0.5 \cdot 10^{-4}}{2h} = 0.8 \cdot 10^{-3}$; thus, the error of computing the derivatives y' does not exceed 0.001.

(b) The truncation error may be neglected, the rounding off error being equal to $\frac{4 \cdot 0.5 \cdot 10^{-4}}{h^2} = 2 \cdot 10^{-2}$; thus, the error of computing the derivatives y'' does not exceed 0.02; hence, the last written digit of the value of y'' is doubtful.

2. (a) The error does not exceed 0.0005.

(b) The error does not exceed 0.005, hence, the last written digit of the value of y'' is doubtful.

6.3

1. (a) The optimal spacing $h = 0.2$; the formula

$$y_0' \approx \frac{1}{h}\left(\frac{\Delta y_{-1} + \Delta y_0}{2} - \frac{1}{6}\,\frac{\Delta^3 y_{-2} + \Delta^3 y_{-1}}{2}\right)$$

yields three correct decimal digits at the points

$$x = 1.4 + 0.1 \cdot k \quad (k = 0,\ 1,\ 2,\ \ldots\ 12).$$

(b) The optimal spacing $h = 0.3$; the formula

$$y_0'' \approx \frac{1}{h^2}\left(\Delta^2 y_{-1} - \frac{1}{12}\,\Delta^4 y_{-2}\right)$$

gives an error not exceeding $2.2 \cdot 10^{-3}$, but is applicable only at the points $x = 1.6 + 0.1 \cdot k$ $(k = 0, 1, 2, \ldots, 8)$. $y''|_{x=1.6} \approx 1.071$.

2. (a) The optimal spacing $h = 0.4$; the formula

$$y'_0 \approx \frac{1}{h}\left(\frac{\Delta y_{-1} + \Delta y_0}{2} - \frac{1}{6}\,\frac{\Delta^3 y_{-2} + \Delta^3 y_{-1}}{2} + \frac{1}{30}\,\frac{\Delta^5 y_{-3} + \Delta^5 y_{-2}}{2}\right)$$

gives an error not exceeding 0.00025, but is applicable only at the points $x = 1.2 + 0.2 \cdot k$ $(k = 0, 1, 2, \ldots, 9)$.

(b) The optimal spacing $h = 0.4$; the formula

$$y''_0 \approx \frac{1}{h^2}\left(\Delta^2 y_{-1} - \frac{1}{12}\,\Delta^4 y_{-2}\right)$$

gives an error not exceeding 0.0013, but is applicable only at the points $x = 1.2 + 0.2 \cdot k$ $(k = 0, 1, 2, \ldots, 9)$. $y''|_{x=1.6} = 4.212$.

CHAPTER 7

7.1

1. -0.995; -1; $\Delta = 0.5 \cdot 10^{-2}$; $\delta = 0.5\%$. **2.** 0.3068, $\Delta = 1.3 \cdot 10^{-5}$. **3.** 101/60, $|R| \leqslant 0.7$. **4.** 38, $|R| \leqslant 6$.

5. (a) 0.213, $|R| \leqslant 0.8 \cdot 10^{-2}$, (b) 0.2112, $|R| \leqslant 0.26 \cdot 10^{-3}$.

6. (a) 0.311, $|R| \leqslant 0.19 \cdot 10^{-2}$, (b) 0.29624, $|R| \leqslant 0.13 \cdot 10^{-4}$.

7. (a) 0.903, $|R| \leqslant 0.19 \cdot 10^{-2}$, (b) 1.05575, $|R| \leqslant 0.13 \cdot 10^{-4}$.

8. (a) 1.828, $|R| \leqslant 2.5 \cdot 10^{-4}$, (b) 1.827847, $|R| \leqslant 2.5 \cdot 10^{-7}$.

9. (a) 0.697, $|R| \leqslant 0.083$, (b) 0.693, $|R| \leqslant 0.033$.

10. 1.898, $|R| \leqslant 0.025$.

7.2

1. 0.69. **2.** 0.79. **3.** 0.88137. **4.** 0.84. **5.** 0.28. **6.** 0.10. **7.** 0.09. **8.** 0.67. **9.** 0.1999. **10.** 0.2722. **11.** 0.9160. **12.** 0.5294. **13.** 0.5422. **14.** 0.4998. **15.** 0.1728. **16.** 0.1157. **17.** 1.0896. **18.** 0.5179. **19.** 0.5061. **20.** 0.364. **21.** 0.521. **22.** 1.078. **23.** 0.671. **24.** 0.513. **25.** 1.178. **26.** 0.707. **27.** 1.176. **28.** 3.148. **29.** 0.322. **30.** 0.684. **31.** 1.854.

32.

a	I_1	I_2	a	I_1	I_2	a	I_1	I_2
0.10	0.1569	1.2038	0.45	0.6876	1.6687	0.80	1.1514	2.3373
0.15	0.2349	1.2605	0.50	0.7590	1.7498	0.85	1.2098	2.4547
0.20	0.3125	1.3201	0.55	0.8289	1.8353	0.90	1.2659	2.5785
0.25	0.3893	1.3829	0.60	0.8972	1.9255	0.95	1.3196	2.7092
0.30	0.4654	1.4490	0.65	0.9637	2.0205	1.00	1.3708	2.8471
0.35	0.5407	1.5185	0.70	1.0283	2.1205	1.05	1.4195	2.9926
0.40	0.6147	1.5916	0.75	1.0909	2.2260	1.10	1.4657	3.1463

33.

a	I_1	I_2	a	I_1	I_2	a	I_1	I_2
0.5	0.0478	0.4606	1.1	0.1193	0.6704	1.6	0.1526	0.9074
0.6	0.0608	0.4941	1.2	0.1283	0.7106	1.7	0.1556	0.9702
0.7	0.0737	0.5276	1.3	0.1362	0.7537	1.8	0.1574	1.0411
0.8	0.0862	0.5615	1.4	0.1429	0.8003	1.9	0.1582	1.1222
0.9	0.0981	0.5964	1.5	0.1484	0.8512	2.0	0.1578	1.2163
1.0	0.1092	0.6325						

34. 0.88208, $R \leqslant \frac{1}{15} \cdot 10^{-5}$.

7.3

1. 0.693146, $|R_4| \leqslant 2.3 \cdot 10^{-5}$. **2.** 0.6931472, $|R_5| \leqslant 1.5 \cdot 10^{-6}$. **3.** 1.21895, $|R_4| \leqslant 5.8 \cdot 10^{-4}$. **4.** 0.402184. **5.** 0.27219801. **6.** 0.272203. **7.** 0.822467.

8.

a	I	a	I	a	I
0.5	0.70684	1.3	0.49066	2.1	0.38722
0.7	0.63423	1.5	0.45824	2.3	0.36973
0.9	0.57638	1.7	0.43082	2.5	0.35451
1.1	0.52941	1.9	0.40739		

9.

a	I	a	I	a	I
0.5	2.29086	1.0	1.55367	1.5	0.85034
0.6	2.15742	1.1	1.39955	1.6	0.74126
0.7	2.01412	1.2	1.24969	1.7	0.64753
0.8	1.86380	1.3	1.10666	1.8	0.57143
0.9	1.70936	1.4	0.97282	1.9	0.51765

7.4

1. 0.0976, $|R| < 0.6 \cdot 10^{-6}$. **2.** 0.9046, $|R| < 10^{-3}$. **3.** 0.052, $|R| < 10^{-3}$. **4.** 0.04493, $|R| < 0.54 \cdot 10^{-4}$. **5.** 1.60543. **6.** 1.852. **7.** 0.6839. **8.** 0.20136. **9.** 1.057. **10.** 0.337. **11.** 0.962. **12.** 0.488. **13.** 0.507. **14.** 0.0119. **15.** 0.6078. **16.** 0.6205. **17.** 0.0709. **18.** 32.8305. **19.** 0.0264. **20.** 0.0980. **21.** 0.1050. **22.** 0.5013 **23.** 0.2225. **24.** 0.3314. **25.** 0.3402. **26.** 0.6042.

7.5

1. *Hint.* Divide the interval of integration into two subintervals: [0; 0.5], [0.5; 1]. 5.24426. **2.** 1.570796 **3.** 0.64041. **4.** 0.785398. **5.** 1.047198. **6.** 11.397928. **7.** 1.570796. **9.** 3.141593.

10.

a \ b	2.00	2.25	2.50	2.75	3.00	3.25	3.50
0.60	0.57999	0.52349	0.47713	0.43838	0.40549	0.37722	0.35266
0.67	0.55432	0.50045	0.45623	0.41925	0.38785	0.36086	0.33741
0.74	0.53003	0.47865	0.43644	0.40114	0.37116	0.34538	0.32296
0.81	0.50704	0.45800	0.41771	0.38399	0.35535	0.33071	0.30928
0.88	0.48528	0.43846	0.39997	0.36775	0.34037	0.31681	0.29633
0.95	0.46466	0.41995	0.38317	0.35236	0.32618	0.30365	0.28405
1.02	0.44514	0.40241	0.36724	0.33779	0.31274	0.29117	0.27241
1.09	0.42664	0.38579	0.35215	0.32397	0.29999	0.27934	0.26137
1.16	0.40911	0.37003	0.33785	0.31087	0.28791	0.26813	0.25091

11. *Hint*. Take into account the pairity of the integrand function. Then Hermite's formula will have the form

$$I=\int_{-1}^{1}\frac{\cos ax\,dx}{(0.3+x^2)\sqrt{1-x^2}}\approx\frac{\pi}{12}\sum_{k=1}^{12}\frac{\cos ax_k}{0.3+x_k^2}=\frac{\pi}{6}\sum_{k=1}^{6}\frac{\cos ax_k}{0.3+x_k^2}.$$

a	2.60	2.64	2.68	2.72	2.76	2.80
I	1.24024	1.16812	1.09718	1.02742	0.95896	0.89172
a	2.84	2.88	2.92	2.96	3.00	
I	0.82588	0.76142	0.69844	0.63696	0.57702	

7.6

1. 0.38. **2.** 0.06270.

3.

a	0.5	0.6	0.7	0.8	0.9	1.0	1.1	1.2	1.3	1.4	1.5
I	0.077	0.071	0.066	0.061	0.058	0.055	0.052	0.050	0.048	0.046	0.045

5. 0.277778. **6.** 0.955933.

7.

a	I
0.5	0.24295
0.6	0.25830
0.7	0.26686
0.8	0.26988
0.9	0.26850
1.0	0.26378
1.1	0.25666
1.2	0.24798
1.3	0.23843
1.4	0.22857
1.5	0.21869

8.

a	I
0.6	0.72302
0.8	0.62692
1.0	0.55411
1.2	0.49687
1.4	0.45061
1.6	0.41240
1.8	0.38028
2.0	0.35288
2.2	0.32922
2.4	0.30857
2.6	0.29038

9.

a	I
1.5	1.31617
1.7	1.31748
1.9	1.31390
2.1	1.30606
2.3	1.29506
2.5	1.28236
2.7	1.27010
2.9	1.26047
3.1	1.25482
3.3	1.25372
3.5	1.25714

10.

a	I
1.0	1.17767
1.2	0.93373
1.4	0.74529
1.6	0.59541
1.8	0.47460
2.0	0.37701
2.2	0.29865
2.4	0.23657
2.6	0.18846
2.8	0.15240
3.0	0.12677

7.7

1. $4\frac{2}{3}=4.667$. **2.** $\ln\frac{25}{24}\approx 0.0408$. **3.** $\frac{\pi}{12}\approx 0.2618$.

4.

a \ b	0.5	0.6	0.7	0.8	0.9	1.0
0.5	0.1840	0.1804	0.1769	0.1737	0.1705	0.1676
0.6	0.1732	0.1699	0.1668	0.1639	0.1611	0.1584
0.7	0.1636	0.1606	0.1578	0.1551	0.1526	0.1501
0.8	0.1549	0.1523	0.1497	0.1473	0.1449	0.1427
0.9	0.1472	0.1447	0.1424	0.1402	0.1380	0.1360
1.0	0.1401	0.1379	0.1358	0.1337	0.1318	0.1299

5.

a \ b	0.0	0.1	0.2	0.3	0.4	0.5
0.0	2.0000	1.6715	1.4410	1.2689	1.1351	1.0276
0.2	1.3518	1.1377	0.9857	0.8714	0.7819	0.7097
0.4	0.9290	0.7872	0.6855	0.6083	0.5475	0.1982
0.6	0.6488	0.5535	0.4843	0.4314	0.3895	0.3553
0.8	0.4603	0.3952	0.3475	0.3107	0.2813	0.2572
1.0	0.3315	0.2864	0.2529	0.2269	0.2060	0.1888

7. 2.19. **8.** π. **9.** 2.538. **10.** $3\pi\sqrt{3}$. **11.** $\frac{\pi}{12}\approx 0.2618$. **12.** $\frac{1}{2}$. **13.** $\frac{1}{2}\ln 2-\frac{5}{16}\approx 0.034$. **14.** $\pi(1-e^{-1})\approx 1.98$.

CHAPTER 8

8.2

1. $y=\frac{1}{2}+\frac{1}{4}x+\frac{1}{4}\frac{x^2}{2!}+\frac{3}{8}\frac{x^3}{3!}+\frac{27}{4}\frac{x^4}{4!}+\dots$.

2. $y=x-\frac{2}{3}x^3+\frac{2}{5}x^5-\frac{2}{7}x^7+\dots$.

3. $y=2(x-1)+2(x-1)^2+\frac{5}{3}(x-1)^3+\frac{7}{4}(x-1)^4+\dots$.

4. $y=1+(x-1)+(x-1)^2+\frac{(x-1)^3}{3!}+\dots$.

5. $y=a\left(1-\frac{x^2}{2!}+\frac{2x^4}{4!}-\frac{9x^6}{6!}+\frac{55x^8}{8!}+\dots\right)$.

6. $y=1+\frac{x^2}{2}-\frac{x^4}{6}+\dots$.

7. $y=x-\frac{x^2}{2}+\frac{2}{3}x^3-\frac{x^4}{4}+\dots$, $z=1-x+x^2-\frac{x^3}{2}+\frac{7}{24}x^4+\dots$.

8. $y=1+x+\frac{x^2}{2}-\frac{x^3}{3}-\frac{x^4}{6}+\dots$, $z=-1+\frac{x^2}{2}+\frac{x^3}{3}+\frac{x^4}{8}+\dots$.

9. $y=x-\frac{1}{2}x^2+\frac{1}{3}x^3+\frac{1}{60}x^6+\frac{17}{1260}x^7+\dots$.

10. $y=1+\frac{1}{2}x+\sum_{n=1}^{\infty}(-1)^n\frac{1\cdot3\dots(2n-1)}{2^{n-1}(2n+2)!}x^{2n+2}$.

11. $y=1-\frac{x}{2!}+\frac{x^2}{4!}-\frac{x^3}{6!}+\dots+(-1)^{n-1}\frac{x^{n-1}}{(2n-2)!}+\dots=\cos\sqrt{x}$.

12. $y=1+\sum_{n=1}^{\infty}(-1)^n\frac{1}{(n!)^2}\left(\frac{x}{2}\right)^{2n}$.

13. $y=1-\frac{x^2}{3!}+\dots+(-1)^n\frac{x^{2n-2}}{(2n-1)!}+\dots=\frac{\sin x}{x}$.

14. (a) $y_1=0.51281$, $y_2=0.52631$, $y_3=0.54052$. (b) $y_1=0.04883$, $y_2=0.09567$, $y_3=0.14088$, $z_1=0.95246$, $z_2=0.90953$, $z_3=0.87096$. (c) $y_1=1.05121$, $y_2=1.10532$, $y_3=1.16004$, $z_1=-0.99875$, $z_2=-0.99466$, $z_3=-0.98757$.

8.3

1. $y_1=\frac{x^3}{3}$, $y_2=\frac{x^3}{3}-\frac{x^7}{63}$, $y_3=\frac{x^3}{3}-\frac{x^7}{63}+\frac{2x^{11}}{189\cdot11}+\frac{x^{15}}{63^2\cdot15}$.

3. $y_1=1+x^2$, $y_2=1+x^2+\frac{2}{3}x^3+\frac{x^5}{5}$, $y_3=1+x^2+\frac{2}{3}x^3+\frac{1}{3}x^4+\frac{1}{5}x^5+\frac{13}{45}x^6+\frac{4}{63}x^7+\frac{1}{20}x^8+\frac{4}{135}x^9+\frac{x^{11}}{275}$.

4. $y_1=\frac{2}{3}x^{3/2}$, $y_2=\frac{2}{3}x^{3/2}+\frac{1}{9}x^4$, $y_3=\frac{2}{3}x^{3/2}+\frac{1}{9}x^4+\frac{x^9}{729}+\frac{8x^{13/2}}{351}$.

5. $y_1=\frac{2}{3}x^{3/2}$, $y_2=\frac{2}{3}x^{3/2}+\frac{4}{21}x^{7/2}$, $y_3=\frac{2}{3}x^{3/2}+\frac{4}{21}x^{7/2}+\frac{8}{231}x^{11/2}$.

6. $y_1=\frac{x^2}{2}$, $y_2=\frac{x^2}{2}+x\sin x-\frac{x^2}{2}\cos x+\cos x$, $y_3=\frac{3}{4}x^2+x\sin x+\left(1-\frac{x^2}{2}\right)\times$
$\times\cos x-\frac{3}{8}x\sin 2x+\frac{2x^2-11}{16}\cos 2x$.

7. $y_1=\frac{x^2}{2}$, $y_2=\frac{x^2}{2}+x\cos x+\left(\frac{x^2}{2}-1\right)\sin x$, $y_3=\frac{3}{4}x^2+x\cos x+\left(\frac{x^2}{2}-1\right)\times$
$\times\sin x+\frac{3}{8}x\sin 2x-\frac{2x^2-11}{16}\cos 2x$.

9. $y_1=x$, $y_2=x+\frac{x^3}{2}$, $y_3=x+\frac{2}{3}x^3+\frac{x^5}{8}$, $z_1=1+\frac{x^2}{2}$, $z_2=1+x^2+\frac{x^4}{8}$,
$z_3=1+x^2+\frac{3}{8}x^4+\frac{x^6}{64}$.

10. $y_1=-x+\frac{x^2}{2}$, $y_2=-x+\frac{x^2}{2}-\frac{x^3}{3}-\frac{x^5}{20}$, $y_3=-x+\frac{x^2}{2}-\frac{x^4}{12}-\frac{x^7}{252}$,
$z_1=1+\frac{x^2}{2}$, $z_2=1+\frac{x^3}{6}$, $z_3=1+\frac{x^3}{6}-\frac{x^4}{12}-\frac{x^6}{120}$.

11. $y_1=-0.5x$, $y_2=-0.5x-0.25x^2$, $y_3=-0.5x-0.25x^2+\frac{0.0125}{3}x^3$, $z_1=0.5$,
$z_2=0.5-0.125x^2$, $z_3=0.5-0.125x^2-\frac{0.125}{3}x^3+0.015625x^4+0.00625x^5$.

12. $y_1=\frac{x^2}{2}$, $y_2=\frac{x^2}{2}+\frac{x^5}{20}$.

13. $y_1=1+x+\frac{x^2}{2}$, $y_2=1+x+x^2+\frac{x^3}{6}$.

14. $y_1=2+x-\frac{2}{3}x^3$, $y_2=2+x+x^2-\frac{2}{3}x^3-\frac{1}{4}x^4$.

15. $y_1=2x-\ln x$, $y_2=2+\ln^2 x$.

16. $y=1-0.01x+1.9999x^2-0.0067x^3-0.0020x^5$.

17. $y=3x^{3/2}+0.225x^4+0.02077x^{13/2}+0.00056x^9$.

18. $y=\frac{x^2}{2}+\frac{1}{7}x^{7/2}+\frac{1}{35}x^5+\frac{2}{455}x^{13/2}$.

8.4

1. $y_1=1$, $y_2=1.005000$, $y_3=1.010025$, $y_4=1.025175$, $y_5=1.045679$, $y_6=1.07.821$, $y_7=1.103976$, $y_8=1.142615$, $y_9=1.188320$, $y_{10}=1.241794$. **2.** $y_1=0$, $y_2=0.001$, $y_3=0.005$, $y_4=0.014002$, $y_5=0.030022$, $y_6=0.055112$, $y_7=0.091416$, $y_8=0.141252$, $y_9=0.207217$, $y_{10}=0.292542$. **3.** $y_1=0.1$, $y_2=0.2001$, $y_3=0.3009$, $y_4=0.40272$, $y_5=0.50920$, $y_6=0.62217$, $y_7=0.74539$, $y_8=0.88428$, $y_9=1.04684$, $y_{10}=1.24547$. **4.** $y_1=1$, $y_2=0.99091$, $y_3=0.97530$, $y_4=0.95520$, $y_5=0.93219$, $y_6=0.90754$, $y_7=0.88188$, $y_8=0.85598$, $y_9=0.83026$, $y_{10}=0.80502$.

5. (a)

β \ x	0.1	0.3	0.5	0.7	0.9	1.0
0.05	0.9050	0.7622	0.6596	0.5820	0.5210	0.4952
0.10	0.9100	0.7726	0.6722	0.5955	0.5318	0.5090
0.15	0.9150	0.7829	0.6848	0.6089	0.5483	0.5224
0.20	0.9200	0.7932	0.6972	0.6221	0.5616	0.5356
0.25	0.9250	0.8035	0.7097	0.6352	0.5718	0.5487
0.30	0.9300	0.8137	0.7220	0.6482	0.5878	0.5616
0.35	0.9350	0.8240	0.7344	0.6612	0.6008	0.5741
0.40	0.9400	0.8342	0.7483	0.6753	0.6114	0.5877

(b)

β \ x	0.1	0.3	0.5	0.7	0.9	1.0
0.1	0.9600	0.8866	0.8219	0.7651	0.7151	0.6923
0.2	0.9700	0.9093	0.8513	0.7979	0.7457	0.7234
0.3	0.9800	0.9319	0.8805	0.8300	0.7827	0.7604
0.4	0.9900	0.9544	0.9094	0.8622	0.8161	0.7939

6.

Variant No. \ x	0.2	0.4	0.6	0.8	1.0
1	0.4094	0.3474	0.3022	0.2675	0.2401
2	0.3709	0.2974	0.2496	0.2157	0.1901
3	0.4096	0.3479	0.3028	0.2684	0.2410
4	0.8189	0.6950	0.6045	0.5352	0.4806
5	0.4100	0.3491	0.3046	0.2705	0.2434
6	1.6377	1.3897	1.2086	1.0702	0.9609
7	1.2288	1.0452	0.9116	0.8087	0.7266
8	0.8198	0.6981	0.6092	0.5412	0.4871
9	1.2212	0.9108	0.7377	0.6254	0.5640
10	1.0948	0.8482	0.7012	0.5988	0.5273
11	1.6380	1.3908	1.2104	1.0727	0.9640
12	1.4836	1.1901	0.9984	0.8627	0.7609
13	0.4109	0.3514	0.3144	0.2768	0.2482
14	1.6390	1.3942	1.2155	1.0789	0.9705
15	1.6397	1.3963	1.2187	1.0825	1.9743
16	2.0496	1.7453	1.5233	1.3531	1.2190
17	1.6430	1.4050	1.2297	1.0940	0.9857
18	3.2793	2.7926	2.4372	2.1647	1.9486
19	0.6151	0.5244	0.4580	0.4073	0.3667
20	1.3451	1.0341	0.8483	0.7232	0.6330

7.

α	β \ x	1.00	1.25	1.50	1.75	2.00
0.2	0.2	0.1992	0.1989	0.1987	0.1984	0.1982
	0.4	0.3890	0.3857	0.3824	0.3791	0.3759
	0.6	0.5587	0.5470	0.5357	0.5248	0.5143
	0.8	0.7018	0.6763	0.6526	0.6306	0.6102
	1.0	0.8164	0.7738	0.7359	0.7020	0.6715
0.4	0.4	0.3917	0.3883	0.3850	0.3817	0.3785
	0.7	0.6490	0.6303	0.6126	0.5959	0.5801
	1.0	0.8541	0.8084	0.7677	0.7314	0.6989
0.6	0.5	0.4881	0.4810	0.4742	0.4675	0.4610
	1.0	0.8934	0.8444	0.8009	0.7621	0.7273
0.8	1.0	0.9346	0.8820	0 8355	0.7940	0.7569
1.0	1.0	0.9776	0.9213	0 8715	0.8272	0.7876

8.

		x	0.4	0.6	0.8	1.0			x	0.4	0.6	0.8	1.0
(a)	y		−0.07841	−0.17202	−0.29507	−0.44005	(b)	y		1.0064	1.0326	1.1042	1.2606
	z		0.96040	0.91200	0.84629	0.76570		z		1.1667	1.3965	1.7662	2.3460

9.

β \ x	0.6	1.0	β \ x	0.6	1.0	β \ x	0.6	1.0
25	0.6277	0.8891	33	0.6243	0.8722	41	0.6210	0.8560
	0.0103	0.5677		0.0433	0.6261		0.0760	0.6828
26	0.6273	0.8869	34	0.6239	0.8701	42	0.6206	0.8540
	0.0144	0.5751		0.0474	0.6333		0.0801	0.6897
27	0.6268	0.8848	35	0.6235	0.8681	43	0.6202	0.8520
	0.0186	0.5825		0.0515	0.6405		0.0841	0.6967
28	0.6264	0.8827	36	0.6231	0.8660	44	0.6197	0.8500
	0.0227	0.5898		0.0556	0.6476		0.0882	0.7036
29	0.6260	0.8805	37	0.6226	0.8640	45	0.6193	0.8481
	0.0268	0.5971		0.0597	0.6547		0.0923	0.7105
30	0.6256	0.8784	38	0.6222	0.8620	46	0.6189	0.8461
	0.0310	0.6044		0.0638	0.6617		0.0963	0.7173
31	0.6252	0.8763	39	0.6218	0.8599	47	0.6185	0.8442
	0.0351	0.6117		0.0679	0.6688		0.1003	0.7241
32	0.6247	0.8743	40	0.6214	0.8579	48	0.6181	0.8423
	0.0392	0.6189		0.0719	0.6758		0.1044	0.7300

10.

α	β \ x	0.5	1.0	α	β \ x	0.5	1.0	α	β \ x	0.5	1.0
2.00	0.25	0.58954	0.57573	2.50	0.25	0.52698	0.50507	3.00	0.25	0.47700	0.45201
		0.55978	0.86672			0.52552	0.80035			0.49801	0.74977
	0.50	0.57307	0.54210		0.50	0.51345	0.47903		0.50	0.46563	0.43105
		0.56056	0.87614			0.52653	0.80958			0.49914	0.75862
	0.75	0.55703	0.51032		0.75	0.50024	0.45418		0.75	0.45449	0.41092
		0.56132	0.88526			0.52752	0.81860			0.50026	0.76731
	1.00	0.54142	0.48035		1.00	0.48733	0.43053		1.00	0.44359	0.39162
		0.56206	0.89406			0.52849	0.82738			0.50136	0.77584
	1.25	0.52622	0.45216		1.25	0.47473	0.40807		1.25	0.43291	0.37317
		0.56279	0.90253			0.52944	0.83592			0.50244	0.78419
2.25	0.25	0.55640	0.53768	2.75	0.25	0.50067	0.47679				
		0.54165	0.83109			0.51105	0.77349				
	0.50	0.54153	0.50826		0.50	0.48830	0.45352				
		0.54256	0.84046			0.51214	0.78254				
	0.75	0.52702	0.48030		0.75	0.47620	0.43124				
		0.54346	0.84957			0.51320	0.79141				
	1.00	0.51287	0.45380		1.00	0.46436	0.40996				
		0.54433	0.85841			0.51424	0.80008				
	1.25	0.49907	0.42875		1.25	0.45279	0.38967				
		0.54518	0.86696			0.51527	0.80855				

8.5

1. 0.0451. **2.** 1.0047. **3.** 0.3181. **4.** 0.0412. **5.** 0.6842. **6.** 3.0042. **7.** 0.1098. **8.** 0.4647. **9.** 0.8032. **10.** 2.1911.

8.6

1.

α	x \ β	−0.3	−0.1	0.1	0.3	0.5
1.00	0.2	0.1611	0.1625	0.1640	0.1656	0.1671
	0.4	0.2579	0.2626	0.2673	0.2722	0.2772
	0.6	0.3101	0.3184	0.3269	0.3358	0.3450
	0.8	0.3327	0.3447	0.3572	0.3704	0.3841
	1.0	0.3374	0.3530	0.3696	0.3872	0.4059
1.25	0.4	0.2428	0.2473	0.2519	0.2566	0.2614
	0.7	0.2898	0.2995	0.3095	0.3200	0.3310
	1.0	0.2897	0.3047	0.3206	0.3375	0.3555
1.50	0.5	0.2465	0.2524	0.2586	0.2649	0.2714
	1.0	0.2540	0.2684	0.2838	0.3003	0.3178
1.75	1.0	0.2319	0.2461	0.2612	0.2774	0.2946
2.00	1.0	0.2252	0.2393	0.2543	0.2704	0.2877

2.

α	x \ β	−0.5	−0.3	−0.1	0.1	0.3
1.00	0.2	0.1802	0.1807	0.1812	0.1817	0.1822
	0.4	0.3228	0.3259	0.3290	0.3322	0.3355
	0.6	0.4324	0.4407	0.4495	0.4586	0.4683
	0.8	0.5146	0.5306	0.5479	0.5666	0.5869
	1.0	0.5748	0.6003	0.6286	0.6603	0.6962
1.25	0.4	0.2690	0.2711	0.2732	0.2754	0.2776
	0.7	0.4085	0.4171	0.4262	0.4358	0.4459
	1.0	0.5050	0.5243	0.5455	0.5689	0.5948
1.50	0.5	0.2769	0.2796	0.2825	0.2854	0.2884
	1.0	0.4499	0.4650	0.4814	0.4993	0.5188
1.75	1.0	0.4053	0.4175	0.4306	0.4446	0.4599
2.00	1.0	0.3687	0.3786	0.3893	0.4007	0.4129

3.

α \ x	0.6	1.0	α \ x	0.6	1.0	α \ x	0.6	1.0
14	−0.0370	0.4814	22	−0.0037	0.5423	30	0.0292	0.6013
15	−0.0328	0.4891	23	0.0004	0.5498	31	0.0333	0.6085
16	−0.0286	0.4968	24	0.0045	0.5572	32	0.0374	0.6158
17	−0.0245	0.5045	25	0.0087	0.5647	33	0.0415	0.6229
18	−0.0203	0.5121	26	0.0128	0.5721	34	0.0456	0.6301
19	−0.0162	0.5197	27	0.0169	0.5794	35	0.0496	0.6372
20	−0.0120	0.5273	28	0.0210	0.5867	36	0.0537	0.6443
21	−0.0079	0.5348	29	0.0251	0.5940	37	0.0578	0.6514

8.7

1. (a) $y(1) = 3.36$, (b) $y(2) = 0.80$, (c) $y(1) = 3.72$, $z(1) = 2.72$.

2.

b	t \ a	1.0	1.4	1.8	2.2	2.6	3.0
1.0	0.1	0.09951	0.07122	0.05543	0.04536	0.03839	0.03327
	0.2	0.19615	0.14124	0.11012	0.09019	0.07635	0.06619
	0.3	0.28759	0.20891	0.16337	0.13396	0.11347	0.09840
1.8	0.1	0.09914	0.07096	0.05522	0.04519	0.03825	0.03315
	0.2	0.19330	0.13915	0.10849	0.08885	0.07522	0.06520
	0.3	0.27844	0.20206	0.15795	0.12950	0.10969	0.09511
2.6	0.1	0.9856	0.7054	0.05490	0.04493	0.03802	0.03295
	0.2	0.18886	0.13591	0.10595	0.08677	0.07345	0.06367
	0.3	0.26433	0.19154	0.14965	0.12267	0.10389	0.09028

3.

b	t \ a	1.0	1.4	1.8	2.2	2.6	3.0
1.0	0.1	0.09934	0.13865	0.17755	0.21595	0.25373	0.29085
	0.2	0.19498	0.26995	0.34225	0.41155	0.47769	0.54065
	0.3	0.28418	0.38933	0.48779	0.57943	0.66452	0.74360
1.4	0.1	0.07117	0.09953	0.12777	0.15587	0.18380	0.21153
	0.2	0.14086	0.19635	0.25103	0.30472	0.35727	0.40858
	0.3	0.20780	0.28831	0.36646	0.44189	0.51438	0.58386
1.8	0.1	0.05542	0.07755	0.09963	0.12166	0.14363	0.16552
	0.2	0.11005	0.15375	0.19714	0.24013	0.28265	0.32464
	0.3	0.16319	0.22744	0.29072	0.35283	0.41359	0.47288

b	t \ a	1.0	1.4	1.8	2.2	2.6	3.0
2.2	0.1	0.04537	0.06350	0.08161	0.099699	0.11776	0.13578
	0.2	0.09025	0.12621	0.16202	0.19764	0.23305	0.26820
	0.3	0.13419	0.18739	0.24014	0.29231	0.34381	0.39456
2.6	0.1	0.03840	0.05376	0.06910	0.08443	0.09974	0.11504
	0.2	0.07647	0.10699	0.13743	0.16777	0.19800	0.22809
	0.3	0.11389	0.15920	0.20427	0.24904	0.29343	0.33741
3.0	0.1	0.03329	0.04660	0.05991	0.07321	0.08650	0.09978
	0.2	0.06634	0.09283	0.11928	9.14568	0.17201	0.19826
	0.3	0.09891	0.13834	0.17762	0.21673	0.25562	0.29427

4.

b	t \ a	1.0	1.4	1.8	2.2	2.6	3.0
1.0	0.1	0.09545	0.093588	0.09177	0.09001	0.08829	0.08661
	0.2	0.18328	0.17623	0.16956	0.16323	0.15723	0.15154
	0.3	0.26515	0.24999	0.23605	0.22321	0.21137	0.20043
1.4	0.1	0.13379	0.13118	0.12863	0.12615	0.12373	0.12138
	0.2	0.25774	0.24774	0.23829	0.22933	0.22085	0.21281
	0.3	0.37467	0.35289	0.33292	0.31457	0.29768	0.28211
1.8	0.1	0.17223	0.16885	0.16556	0.16236	0.15925	0.15621
	0.2	0.33287	0.31985	0.30755	0.29591	0.28489	0.27445
	0.3	0.48629	0.45753	0.43125	0.40716	0.38503	0.36467
2.2	0.1	0.21076	0.20662	0.20258	0.19866	0.19484	0.19112
	0.2	0.40869	0.39257	0.37735	0.36296	0.34936	0.33648
	0.3	0.60006	0.56397	0.53107	0.50100	0.47344	0.44813
2.6	0.1	0.24938	0.24447	0.23969	0.23503	0.23051	0.22610
	0.2	0.48520	0.46590	0.44769	0.43050	0.41426	0.39889
	0.3	0.71605	0.67225	0.63243	0.59613	0.56294	0.53250
3.0	0.1	0.28810	0.28241	0.27688	0.27149	0.26625	0.26115
	0.2	0.56241	0.53985	0.51859	0.49853	0.47960	0.46170
	0.3	0.83436	0.78243	0.73537	0.69258	0.65354	0.61782

5.

t \ a	1.0	1.4	1.8	2.2	2.6	3.0
0.1	0.09935	0.09907	0.09874	0.09835	0.09792	0.09744
0.2	0.19504	0.19290	0.19041	0.18757	0.18439	0.18090
0.3	0.28411	0.27743	0.26974	0.26111	0.25162	0.24135

6.

b	t \ a	1.0	1.4	1.8	2.2	2.6	3.0
1.0	0.1	0.092578	0.09063	0.08870	0.08677	0.08486	0.08295
	0.2	0.17087	0.16346	0.15616	0.14900	0.14200	0.13517
	0.3	0.23610	0.22043	0.20529	0.19076	0.17691	0.16383
1.8	0.1	0.094411	0.092442	0.09048	0.08853	0.086591	0.08464
	0.2	0.17759	0.16999	0.16252	0.15519	0.14801	0.14101
	0.3	0.24999	0.23378	0.21812	0.20309	0.18876	0.17522
2.6	0.1	0.09629	0.094297	0.09231	0.09034	0.08837	0.08642
	0.2	0.18466	0.17688	0.16922	0.16171	0.15435	0.14719
	0.3	0.26497	0.24821	0.23200	0.21644	0.20160	0.18757

7.

t \ a	1.0	1.4	1.8	2.2	2.6	3.0
0.1	0.09548	0.06910	0.05414	0.04450	0.03778	0.03282
0.2	0.18364	0.13437	0.10593	0.08742	0.07441	0.06478
0.3	0.26679	0.19694	0.15604	0.12919	0.11022	0.09610

8.

t \ a	1.0	1.4	1.8	2.2	2.6	3.0
0.1	0.09999	0.10013	0.10027	0.10040	0.10053	0.10067
0.2	0.19999	0.20105	0.20211	0.20318	0.20425	0.20534
0.3	0.29992	0.30345	0.30702	0.31065	0.31432	0.31806

9.

a	t \ b	2.0	2.5	3.0	3.5	4.0	4.5
2.0	0.1	1.21652	1.26593	1.31473	1.36319	1.41143	1.45953
		0.23511	0.21685	0.20478	0.19623	0.18989	0.18503
	0.2	1.38034	1.47783	1 57325	1.66758	1.76131	1.85474
		0.40685	0.37310	0 35110	0.33579	0.32462	0.31617
	0.3	1 50534	1.65038	1.79202	1.93219	2.07196	2 21198
		0.57712	0.52953	0.49901	0.47810	0.46306	0.45184
2.5	0.1	1.20706	1.25750	1.30698	1.35593	1.40454	1.45292
		0.25570	0.23271	0.21750	0.20675	0.19878	0.19267
	0.2	1.35237	1.45342	1.55137	1.64762	1.74291	1.83767
		0.44353	0.40042	0 37241	0 35297	0.33881	0.32812
	0.3	1.46127	1.61251	1.75917	1.90358	2.04710	2.19056
		0.62791	0.56626	0.52699	0.50020	0.48101	0.46673
3.0	0.1	1.19663	1.24829	1.29861	1.34816	1.39722	1.44597
		0.27661	0.24876	0.23036	0.21737	0.20775	0.20037
	0.2	1.32428	1 42877	1.52938	1.62773	1.72475	1.82099
		0.48152	0.42852	0.39423	0.37050	0.35325	0.34025
	0.3	1.42632	1.58104	1.73317	1.87990	2.02700	2.17383
		0.68102	0.60438	0 55583	0.52287	0.49934	0.48188
3.5	0.1	1.18546	1.23847	1.28973	1.33996	1.38955	1.43872
		0.29787	0.26504	0.24338	0.22811	0.21680	0.20814
	0.2	1.29878	1.40561	1.50847	1.60880	1.70753	1.80528
		0.52081	0.45744	0.41659	0.38840	0.36796	0.35259
	0.3	1.40987	1.56174	1.71345	1.86399	2.01380	2.16346
		0.73566	0.64359	0.58541	0.54603	0.51800	0.49727

10.

a	t \ b	2.0	2.5	3.0	3.5	4.0	4.5
2.0	0.1	1.13650	1.13587	1.13527	1.13469	1.13413	1.13358
		0.64576	0.63125	0.61745	0.60430	0.59176	0.57979
	0.2	1.25010	1.24782	1.24569	1.24369	1.24182	1.24006
		0.77826	0.74628	0.71725	0.69079	0.66657	0.64431
	0.3	1.33743	1.33274	1.32853	1.32471	1.32124	1.31806
		0.89249	0.84199	0.79807	0.75951	0.72536	0.69490
2.5	0.1	1.09863	1.09805	1.09748	1.09693	1.09639	1.09588
		0.64408	0.62961	0.61584	0.60271	0.59020	0.57825
	0.2	1.17473	1.17258	1.17057	1.16870	1.16695	1.16530
		0.77195	0.74020	0.71138	0.68510	0.66104	0.63893
	0.3	1.23528	1.23076	1.22671	1.22306	1.21975	1.21673
		0.88030	0.83046	0.78709	0.74900	0.71527	0.68518
3.0	0.1	1.05806	1.05749	1.05694	1.05640	1.05589	1.05539
		0.64212	0.62768	0.61394	0.60084	0.58836	0.57643
	0.2	1.10686	1.10476	1.10281	1.10099	1.09928	1.09768
		0.76561	0.73409	0.70548	0.67939	0.65550	0.63354
	0.3	1.15086	1.14640	1.14240	1.13881	1.13556	1.13260
		0.86917	0.81995	0.77711	0.73947	0.70613	0.67664
3.5	0.1	1.02094	1.02038	1.01983	1.01930	1.01879	1.01829
		0.64023	0.62582	0.61210	0.59904	0.58658	0.57468
	0.2	1.04931	1.04722	1.04527	1.04345	1.04175	1.04016
		0.75993	0.72863	0.70021	0.67429	0.65055	0.62873
	0.3	1.08182	1.07733	1.07333	1.06974	1.06649	1.06353
		0.85962	0.81093	0.76855	0.73130	0.69830	0.66885

11.

a	t \ b	2.0	2.5	3.0	3.5	4.0	4.5
2.0	0.1	0.68427	0.68235	0.68134	0.68150	0.68300	0.68566
		1.01614	0.98056	0.96104	0.96312	0.99197	1.04688
	0.2	0.89997	0.89232	0.88804	0.88835	0.89480	0.90792
		1.04795	0.98208	0.94347	0.94317	1.00401	1.14544
	0.3	1.15392	1.13666	1.12668	1.12682	1.14280	1.17954
		1.09630	1.00440	0.94865	0.94485	1.04729	1.31929
2.5	0.1	0.69489	0.69296	0.69194	0.69210	0.69361	0.69629
		1.01660	0.98103	0.96152	0.96362	0.99251	1.04745
	0.2	0.92661	0.91886	0.91454	0.91487	0.92141	0.93469
		1.05002	0.98417	0.94565	0.94557	1.00680	1.14852
	0.3	1.20364	1.18605	1.17591	1.17612	1.19251	1.22991
		1.10154	1.00968	0.95423	0.95121	1.05553	1.32768
3.0	0.1	0.70464	0.70270	0.70167	0.70184	0.70335	0.70604
		1.01703	0.98146	0.96195	0.96407	0.99300	1.04797
	0.2	0.95146	0.94363	0.93927	0.93961	0.94624	0.95966
		1.05194	0.98610	0.94766	0.94778	1.00938	1.15135
	0.3	1.25072	1.23283	1.22254	1.22282	1.23959	1.27759
		1.10645	1.01463	0.95945	0.95717	1.06325	1.33543
3.5	0.1	0.71373	0.71178	0.71075	0.71092	0.71244	0.71515
		1.01742	0.98185	0.96236	0.96451	0.99346	1.04845
	0.2	0.97498	0.96707	0.96266	0.96302	0.96973	0.98329
		1.05374	0.98791	0.94956	0.94986	1.01180	1.15400
	0.3	1.29588	1.27771	1.26728	1.26763	1.28476	1.32332
		1.11110	1.01933	0.96441	0.96284	1.07057	1.34271

12.

a	t \ b	2.0	2.5	3.0	3.5	4.0	4.5
2.0	0.1	0.53403	0.53370	0.53337	0.53305	0.53274	0.53242
		1.16116	1.17512	1.18908	1.20302	1.21695	1.23087
	0.2	0.57946	0.57832	0.57721	0.57615	0.57512	0.57413
		1.34819	1.37964	1.41101	1.44229	1.47349	1.50461
	0.3	0.63792	0.63584	0.63388	0.63205	0.63033	0.62871
		1.56754	1.62119	1.67456	1.72767	1.78054	1.83319
2.5	0.1	0.53650	0.53618	0.53585	0.53553	0.53522	0.53490
		1.16134	1.17535	1.18935	1.20333	1.21731	1.23127
	0.2	0.58932	0.58818	0.58708	0.58601	0.58498	0.58399
		1.34974	1.38158	1.41333	1.44500	1.47658	1.50808
	0.3	0.66005	0.65798	0.65604	0.65421	0.65250	0.65090
		1.57329	1.62835	1.68314	1.73766	1.79194	1.84600
3.0	0.1	0.53898	0.53866	0.53833	0.53801	0.53769	0.53738
		1.16152	1.17557	1.18961	1.20365	1.21767	1.23168
	0.2	0.59918	0.59804	0.59694	0.59588	0.59485	0.59386
		1.35130	1.38353	1.41567	1.44772	1.47969	1.51158
	0.3	0.68219	0.68012	0.67819	0.67638	0.67468	0.67309
		1.57916	1.63567	1.69191	1.74787	1.80360	1.85909
3.5	0.1	0.54146	0.54114	0.54081	0.54049	0.54017	0.53986
		1.16169	1.17580	1.18988	1.20396	1.21803	1.23208
	0.2	0.60903	0.60790	0.60680	0.60574	0.60471	0.60373
		1.35288	1.38551	1.41804	1.45048	1.48284	1.51512
	0.3	0.70432	0.70227	0.70035	0.69855	0.69687	0.69529
		1.58515	1.64315	1.70086	1.75831	1.81550	1.87246

13.

a	t \ b	2.0	2.5	3.0	3.5	4.0	4.5
2.0	0.1	0.95749	0.96095	0.96427	0.96745	0.97052	0.97347
		0.59814	0.59826	0.59834	0.59849	0.59860	0.59871
	0.2	0.95840	0.96896	0.97889	0.98828	0.99719	1.00566
		0.69837	0.69914	0.69988	0.70057	0.70124	0.70188
	0.3	0.99197	1.01110	1.02894	1.04564	1.06136	1.07621
		0.80785	0.80993	0.81189	0.81375	0.81552	0.81720
2.5	0.1	0.95835	0.96178	0.96506	0.96822	0.97125	0.97418
		0.59854	0.59866	0.59878	0.59889	0.59900	0.59911
	0.2	0.96271	0.97299	0.98268	0.99186	1.00057	1.00887
		0.70157	0.70230	0.70301	0.70368	0.70432	0.70494
	0.3	1.00175	1.02016	1.03737	1.05354	1.06878	1.08320
		0.81790	0.81984	0.82168	0.82342	0.82508	0.82666
3.0	0.1	0.95938	0.96276	0.96600	0.96912	0.97212	0.97501
		0.59904	0.59916	0.59927	0.59938	0.59949	0.59959
	0.2	0.96757	0.97755	0.98698	0.99592	1.00442	1.01253
		0.70534	0.70604	0.70672	0.70734	0.70797	0.70856
	0.3	1.01211	1.03007	1.04663	1.06223	1.07697	1.09094
		0.82948	0.83128	0.83298	0.83461	0.83615	0.83764
3.5	0.1	0.96055	0.96388	0.96708	0.97015	0.97311	0.97597
		0.59961	0.59973	0.59984	0.59995	0.60006	0.60016
	0.2	0.97284	0.98251	0.99166	1.00036	1.00864	1.01655
		0.70964	0.71031	0.71095	0.71156	0.71214	0.71271
	0.3	1.02359	1.04051	1.05642	1.07144	1.08567	1.09920
		0.84234	0.84400	0.84558	0.84709	0.84853	0.84991

8.8

1. (a) $y(0.5) = 1.80$, (b) $y(1) = 3.15$, (c) $y(0.5) = 0.14$, (d) $y(0.5) = 3.15$, $z(0.5) = -3.15$, (e) $y(0.5) = 0.55$, $z(0.5) = -0.18$.

2.

x	0.3	0.5	0.8	1.0	x	1.3	1.5	1.8	2.0
(a)	0.7645	0.6511	0.5116	0.4388	(c)	−0.2676	−0.4242	−0.6450	−0.7895
(b)	0.7408	0.6065	0.4493	0.3679					

3.

b	t \ a	1.0	1.4	1.8	2.2	2.6	3.0
1.0	0.4	0.37223	0.27329	0.21451	0.17618	0.14935	0.12956
	0.5	0.44921	0.33360	0.26298	0.21638	0.18360	0.15936
1.8	0.4	0.35181	0.25759	0.20199	0.16582	0.14054	0.12191
	0.5	0.41189	0.30414	0.23923	0.19666	0.16678	0.14472
2.6	0.4	0.32070	0.23388	0.18314	0.15026	0.12731	0.11042
	0.5	0.35561	0.26050	0.20431	0.16774	0.14217	0.12332

4.

b	t \ a	1.0	1.4	1.8	2.2	2.6	3.0
1.0	0.4	0.36556	0.49545	0.61391	0.72161	0.81974	0.90955
	0.5	0.43883	0.58896	0.72299	0.84281	0.95059	1.04831
1.4	0.4	0.27107	0.37403	0.47236	0.56564	0.65376	0.73687
	0.5	0.33012	0.45294	0.56841	0.67624	0.77668	0.87024
1.8	0.4	0.21424	0.29771	0.37914	0.45818	0.53459	0.60822
	0.5	0.26278	0.36397	0.46171	0.55551	0.64515	0.73058
2.2	0.4	0.17679	0.24645	0.31512	0.38256	0.44861	0.51313
	0.5	0.21774	0.30293	0.38637	0.46770	0.54668	0.62316
2.6	0.4	0.15038	0.20998	0.26903	0.32742	0.38502	0.44174
	0.5	0.18570	0.25896	0.33124	0.40234	0.47206	0.54027
3.0	0.4	0.13079	0.18279	0.23448	0.28576	0.33658	0.38687
	0.5	0.16181	0.22594	0.28949	0.35233	0.41432	0.47538

5.

b	t \ a	1.0	1.4	1.8	2.2	2.6	3.0
1.0	0.4	0.34217	0.31623	0.29313	0.27247	0.25393	0.23724
	0.5	0.41512	0.37593	0.34217	0.31289	0.28733	0.26491
1.4	0.4	0.48648	0.44860	0.41506	0.38522	0.35855	0.33463
	0.5	0.59444	0.53616	0.48649	0.44374	0.40668	0.37433
1.8	0.4	0.63541	0.58458	0.53985	0.50024	0.46500	0.43350
	0.5	0.78230	0.70262	0.63542	0.57809	0.52871	0.48584
2.2	0.4	0.78923	0.72435	0.66759	0.61761	0.57332	0.53388
	0.5	0.97944	0.87574	0.78924	0.71609	0.65352	0.59949
2.6	0.4	0.94822	0.86808	0.79843	0.73742	0.68359	0.63584
	0.5	1.18669	1.05599	0.94824	0.85792	0.78122	0.71537
3.0	0.4	1.11269	1.01599	0.93250	0.85975	0.79586	0.73939
	0.5	1.40497	1.24391	1.11272	1.00378	0.91194	0.83355

6.

t \ a	1.0	1.4	1.8	2.2	2.6	3.0
0.4	0.36455	0.35014	0.33381	0.31583	0.29642	0.27586
0.5	0.43534	0.41002	0.38190	0.35155	0.31952	0.28638

7.

b	t \ a	1.0	1.4	1.8	2.2	2.6	3.0
1.0	0.4	0.28977	0.26394	0.23958	0.21692	0.19617	0.17755
	0.5	0.33352	0.29650	0.26268	0.23258	0.20668	0.18543
1.8	0.4	0.31256	0.28563	0.26022	0.23657	0.21490	0.19542
	0.5	0.36657	0.32772	0.29221	0.26056	0.23325	0.21076
2.6	0.4	0.33776	0.30966	0.28314	0.25844	0.23578	0.21539
	0.5	0.40402	0.36321	0.32590	0.29258	0.26378	0.23997

8.

t \ a	1.0	1.4	1.8	2.2	2.6	3.0
0.4	0.34690	0.25782	0.20508	0.17022	0.14549	0.12702
0.5	0.42579	0.31798	0.25364	0.21093	0.18051	0.15775

9.

t \ a	1.0	1.4	1.8	2.2	2.6	3.0
0.4	0.39967	0.40789	0.41632	0.42494	0.43378	0.44284
0.5	0.49900	0.51471	0.53100	0.54789	0.56540	0.58355

10.

a	t \ b	2.0	2.5	3.0	3.5	4.0	4.5
2.0	0.4	1.60942	1.80319	1.99356	2.18344	2.37452	2.56783
		0.75058	0.69019	0.65208	0.62633	0.60805	0.59457
	0.5	2.71222	1.95814	2.20321	2.45094	2.70332	2.96136
		0.92917	0.85677	0.81178	0.78178	0.76073	0.74536
2.5	0.4	1.56640	1.76532	1.96331	2.16049	2.35854	2.55853
		0.81378	0.73478	0.68536	0.65219	0.62875	0.61153
	0.5	1.69215	1.94406	2.19860	2.45662	2.71909	2.98644
		1.00248	0.90757	0.84909	0.81038	0.78337	0.76374
3.0	0.4	1.55043	1.75005	1.95134	2.15343	2.35688	2.56232
		0.87920	0.78082	0.71954	0.67861	0.64981	0.62874
	0.5	1.73662	1.97627	2.23012	2.49167	2.75888	3.03065
		1.07599	0.95902	0.88689	0.83930	0.80622	0.78226
3.5	0.4	1.58301	1.76695	1.96387	2.16627	2.37203	2.58055
		0.94404	0.82723	0.75413	0.70534	0.67110	0.64610
	0.5	1.84136	2.05502	2.29670	2.55306	2.81791	3.08795
		1.14517	1.00923	0.92433	0.86812	0.82903	0.80078

11.

a	t \ b	2.0	2.5	3.0	3.5	4.0	4.5
2.0	0.4	1 40782	1.39991	1.39308	1.38710	1.38182	1.37711
		0.98740	0.91911	0.86185	0.81310	0.77105	0.73437
	0.5	1.47160	1.45919	1.44891	1.44022	1.43277	1.42628
		1.06510	0.98085	0.91233	0.85539	0.80725	0.76594
2.5	0.4	1.28867	1.28084	1.27412	1.26826	1.26311	1.25853
		0.96970	0.90263	0.84637	0.79844	0.75709	0.72102
	0.5	1.34232	1.32961	1.31918	1.31045	1.30301	1.29658
		1.04307	0.96060	0.89348	0.83769	0.79049	0.75000
3.0	0.4	1.19431	1.18645	1.17972	1.17389	1.16878	1.16426
		0.95448	0.88850	0.83312	0.78592	0.74519	0.70965
	0.5	1.24233	1.22921	1.21856	1.20972	1.20223	1.19581
		1.02483	0.94388	0.87795	0.82313	0.77675	0.73693
3.5	0.4	1.11857	1.11057	1.10379	1.09792	1.09280	1.08827
		0.94174	0.87667	0.82204	0.77547	0.73526	0.70016
	0.5	1.16310	1.14945	1.13848	1.12946	1.12188	1.11541
		1.00981	0.93011	0.86518	0.81116	0.76544	0.72618

12.

a	t \ b	2.0	2.5	3.0	3.5	4.0	4.5
2.0	0.4	1.45383	1.42292	1.40469	1.40434	1.43680	1.51570
		1.16303	1.04841	0.97820	0.97321	1.41105	1.54772
	0.5	1.80870	1.75982	1.73074	1.73001	1.79072	1.92798
		1.25132	1.11675	1.03548	1.03620	1.31848	1.75688
2.5	0.4	1.53580	1.50408	1.48545	1.48533	1.51910	1.59958
		1.17360	1.05910	0.98960	0.98677	1.16086	1.56161
	0.5	1.93485	1.88432	1.85446	1.85436	1.91855	2.05818
		1.27030	1.13614	1.05655	1.06261	1.35978	1.77489
3.0	0.4	1.61460	1.58212	1.56313	1.56324	1.59823	1.68016
		1.18361	1.06925	1.00046	0.99971	1.17962	1.57445
	0.5	2.05790	2.00582	1.97524	1.97578	2.04324	2.18498
		1.28856	1.15487	1.07700	1.08841	1.39862	1.79182
3.5	0.4	1.69119	1.65800	1.63867	1.63899	1.67516	1.75842
		1.19323	1.07902	1.01092	1.01221	1.19761	1.58648
	0.5	2.17908	2.12553	2.09429	2.09547	2.16605	2.30971
		1.30632	1.17315	1.09707	1.11387	1.43544	1.80798

13.

a	t \ b	2.0	2.5	3.0	3.5	4.0	4.5
2.0	0.4	0.71165	0.70884	0.70628	0.70395	0.70183	0.69989
		1.82771	1.90995	1.99162	2.07277	2.15340	2.23372
	0.5	0.80286	0.79966	0.79683	0.79432	0.79207	0.79005
		2.13997	2.25966	2.37840	2.49630	2.61345	2.72994
2.5	0.4	0.75103	0.74825	0.74573	0.74344	0.74135	0.73945
		1.84285	1.92882	2.01421	2.09908	2.18346	2.26742
	0.5	0.86456	0.86144	0.85868	0.85623	0.85401	0.85207
		2.17316	2.30104	2.42794	2.55397	2.67924	2.80383
3.0	0.4	0.79042	0.78768	0.78519	0.78294	0.78089	0.77903
		1.85852	1.94836	2.03751	2.12632	2.21454	2.30233
	0.5	0.92629	0.92325	0.92056	0.91818	0.91604	0.91412
		2.20801	2.34450	2.47998	2.61457	2.74838	2.88149
3.5	0.4	0.82982	0.82712	0.82467	0.82246	0.82045	0.81862
		1.87472	1.96857	2.06181	2.15450	2.24670	2.33844
	0.5	0.98805	0.98508	0.98247	0.98014	0.98706	0.97619
		2.24453	2.39004	2.53453	2.67810	2.82087	2.96293

14.

a	t \ b	2.0	2.5	3.0	3.5	4.0	4.5
2.0	0.4	1.05138	1.07983	1.10617	0.13072	1.15370	1.17531
		0.93160	0.93564	0.93943	1.94302	0.94641	0.94963
	0.5	1.13213	1.17028	1.20546	1.23812	1.26859	1.29717
		1.07299	1.07961	1.08582	1.09166	1.09719	1.10243
2.5	0.4	1.06798	1.09509	1.12031	1.14388	1.16601	1.18688
		0.95334	0.95701	0.96048	0.96377	0.96689	0.96986
	0.5	1.15042	1.19252	1.22597	1.25715	1.28635	1.31381
		1.11145	1.11736	1.12293	1.12819	1.13319	1.13795
3.0	0.4	1.08558	1.11136	1.13544	1.15802	1.17929	1.19940
		0.97782	0.98116	0.98436	0.98731	0.99017	0.99290
	0.5	1.18166	1.21578	1.24755	1.27727	1.30519	1.33153
		1.15103	1.15930	1.16429	1.16903	1.17354	1.17784
3.5	0.4	1.10363	1.12814	1.15112	1.17273	1.19314	1.21249
		1.00450	1.00752	1.01039	1.01313	1.01574	1.01824
	0.5	1.20714	1.23940	1.26956	1.29787	1.32456	1.34981
		1.19976	1.20447	1.20895	1.21321	1.21728	1.22118

15. (a) $y(0.2) = 0.55691$, (b) $y(0.2) = 0.18498$, $z(0.2) = 0.83644$, (c) $y(0.2) = 1.21706$, $z(0.2) = -0.97715$.

8.9

1.

α \ x	1.3	1.5	1.8	2.0	α \ x	1.3	1.5	1.8	2.0
0.50	0.6761	0.5017	0.3285	0.2550	3.25	0.7533	0.6346	0.5022	0.4356
0.75	0.7046	0.5468	0.3803	0.3048	3.50	0.7544	0.6368	0.5056	0.4396
1.00	0.7197	0.5725	0.4129	0.3377	3.75	0.7554	0.6387	0.5087	0.4432
1.25	0.7291	0.5892	0.4353	0.3612	4.00	0.7562	0.6403	0.5114	0.4464
1.50	0.7355	0.6008	0.4516	0.3787	4.25	0.7570	0.6418	0.5138	0.4492
1.75	0.7401	0.6094	0.4640	0.3923	4.50	0.7576	0.6432	0.5159	0.4518
2.00	0.7436	0.6160	0.4737	0.4032	4.75	0.7582	0.6444	0.5179	0.4541
2.25	0.7464	0.6213	0.4816	0.4120	5.00	0.7588	0.6454	0.5196	0.4562
2.50	0.7486	0.6255	0.4881	0.4194	5.25	0.7593	0.6464	0.5213	0.4581
2.75	0.7504	0.6291	0.4935	0.4256	5.50	0.7597	0.6473	0.5227	0.4598
3.00	0.7520	0.6320	0.4982	0.4310					

2.

α \ x	0.3	0.5	0.8	1.0	α \ x	0.3	0.5	0.8	1.0
0.20	2.4329	3.2747	4.2835	4.7477	0.31	1.9086	2.4241	3.0108	3.2547
0.21	2.3626	3.1606	4.1127	4.5473	0.32	1.8788	2.3758	2.9384	3.1699
0.22	2.2986	3.0568	3.9574	4.3652	0.33	1.8509	2.3304	2.8705	3.0902
0.23	2.2402	2.9620	3.8156	4.1989	0.34	1.8245	2.2877	2.8066	3.0152
0.24	2.1867	2.8752	3.6857	4.0464	0.35	1.7997	2.2474	2.7463	2.9445
0.25	2.1374	2.7953	3.5661	3.9062	0.36	1.7762	2.2093	2.6894	2.8777
0.26	2.0920	2.7215	3.4558	3.7767	0.37	1.7540	2.1733	2.6355	2.8145
0.27	2.0499	2.6532	3.3536	3.6569	0.38	1.7330	2.1392	2.5845	2.7547
0.28	2.0108	2.5898	3.2587	3.5456	0.39	1.7131	2.1069	2.5361	2.6979
0.29	1.9744	2.5308	3.1703	3.4419	0.40	1.6941	2.0761	2.4901	2.6440
0.30	1.9404	2.4757	3.0879	3.3452					

8.10

1.

a	h \ x	0.05	0.10	0.15	0.20	0.25
1.0	1.0	0.05110	0.10377	0.15706	0.21005	0.26193
	1.4	0.05162	0.10590	0.16196	0.21856	0.27580
	1.8	0.05214	0.10809	0.16705	0.22816	0.29063
	2.2	0.05268	0.11034	0.17235	0.23798	0.30650
	2.6	0.05322	0.11265	0.17788	0.24834	0.32349
	3.0	0.05376	0.11503	0.18363	0.25929	0.34168
1.4	1.0	0.05102	0.10318	0.15510	0.20557	0.25360
	1.4	0.05154	0.10530	0.15995	0.21424	0.26715
	1.8	0.05207	0.10748	0.16499	0.22338	0.28164
	2.2	0.05260	0.10972	0.17024	0.23304	0.29714
	2.6	0.05314	0.11202	0.17571	0.24324	0.31375
	3.0	0.05368	0.11438	0.18141	0.25401	0.33155
1.8	1.0	0.05094	0.10249	0.15285	0.20050	0.24436
	1.4	0.05145	0.10460	0.15764	0.20901	0.25757
	1.8	0.05198	0.10677	0.16263	0.21801	0.27171
	2.2	0.05251	0.10900	0.16782	0.22749	0.28686
	2.6	0.05305	0.11128	0.17324	0.23751	0.30308
	3.0	0.05359	0.11364	0.17887	0.24810	0.32048
2.2	1.0	0.05084	0.10170	0.15032	0.19490	0.23440
	1.4	0.05135	0.10380	0.15505	0.20326	0.24727
	1.8	0.05188	0.10596	0.15998	0.21208	0.26106
	2.2	0.05241	0.10817	0.16512	0.22140	0.27584
	2.6	0.05294	0.11045	0.17047	0.23125	0.29168
	3.0	0.05349	0.11279	0.17604	0.24165	0.30868
2.6	1.0	0.05072	0.10082	0.14753	0.18887	0.22389
	1.4	0.05124	0.10291	0.15221	0.19706	0.23643
	1.8	0.05176	0.10505	0.15708	0.20572	0.24987
	2.2	0.05229	0.10725	0.16215	0.21486	0.26429
	2.6	0.05283	0.10952	0.16744	0.22453	0.27975
	3.0	0.05337	0.11184	0.17295	0.23474	0.29635
3.0	1.0	0.05060	0.09986	0.14451	0.18247	0.21304
	1.4	0.05111	0.10193	0.14912	0.19050	0.22527
	1.8	0.05163	0.10406	0.15393	0.19898	0.23834
	2.2	0.05216	0.10624	0.15894	0.20795	0.25240
	2.6	0.05270	0.10849	0.16416	0.21743	0.26749
	3.0	0.05324	0.11080	0.16960	0.22746	0.28370

2.

a	b \ x	0.05	0.10	0.15	0.20	0.25
1.0	1.0	0.01886	0.03864	0.05937	0.08104	0.10367
	1.4	0.01905	0.03944	0.06123	0.08448	0.10925
	1.8	0.01924	0.04026	0.06317	0.08811	0.11522
	2.2	0.01944	0.04110	0.06519	0.09195	0.12162
	2.6	0.01964	0.04196	0.06729	0.09600	0.12848
	3.0	0.01984	0.04285	0.06949	0.10029	0.13584
1.4	1.0	0.01264	0.02691	0.03983	0.05441	0.06967
	1.4	0.01277	0.02645	0.04108	0.05672	0.07342
	1.8	0.01290	0.02700	0.04238	0.05916	0.07744
	2.2	0.01303	0.02756	0.04374	0.06174	0.08174
	2.6	0.01316	0.02814	0.04515	0.06446	0.08636
	3.0	0.01330	0.02873	0.04662	0.06734	0.09131
1.8	1.0	0.00847	0.01737	0.02672	0.03652	0.04678
	1.4	0.00856	0.01773	0.02755	0.03806	0.04930
	1.8	0.00865	0.01810	0.02843	0.03970	0.05200
	2.2	0.00874	0.01848	0.02934	0.04143	0.05489
	2.6	0.00882	0.01887	0.03028	0.04326	0.05799
	3.0	0.00892	0.01927	0.03127	0.04519	0.06132
2.2	1.0	0.00568	0.01165	0.01792	0.02449	0.03140
	1.4	0.00574	0.01189	0.01848	0.02554	0.03309
	1.8	0.00580	0.01214	0.01906	0.02663	0.03490
	2.2	0.00586	0.01239	0.01967	0.02779	0.03684
	2.6	0.00592	0.01265	0.02031	0.02902	0.03892
	3.0	0.00598	0.01292	0.02097	0.03032	0.04115
2.6	1.0	0.00381	0.00781	0.01201	0.01643	0.02106
	1.4	0.00385	0.00797	0.01239	0.01712	0.02220
	1.8	0.00388	0.00814	0.01278	0.01786	0.02341
	2.2	0.00392	0.00830	0.01319	0.01864	0.02471
	2.6	0.00396	9.00848	0.01362	0.01946	0.02611
	3.0	0.00401	0.00866	0.01406	0.02033	0.02761
3.0	1.0	0.00255	0.00523	0.00805	0.01101	0.01412
	1.4	0.00258	0.00534	0.00831	0.01148	0.01489
	1.8	0.00260	0.00545	0.00857	0.01198	0.01570
	2.2	0.00263	0.00557	0.00884	0.01250	0.01657
	2.6	0.00266	0.00568	0.00913	0.01305	0.01751
	3.0	0.00268	0.00580	0.00943	0.01363	0.01852

3.

a	x			
	0.05	0.10	0.15	0.20
1.8	0.99775	0.99104	0.97995	0.96464
2.0	0.99750	0.99005	0.97775	0.96079

4.

a	x	k			
		1	2	3	4
1.0	0.05	−0.00119	−0.0004	0.00000	0.00000
	0.10	−0.00455	−0.00033	−0.00002	0.00000
	0.15	−0.00982	−0.00110	−0.00013	−0.00002
	0.20	−0.01678	−0.00257	−0.00040	−0.00006
1.3	0.05	−0.00153	−0.00005	0.00000	0.00000
	0.10	−0.00577	−0.00043	−0.00003	0.00000
	0.15	−0.01232	−0.00142	−0.00016	−0.00002
	0.20	−0.02086	−0.00330	−0.00051	−0.00008
1.6	0.05	−0.00185	−0.00007	0.00000	0.00000
	0.10	−0.00693	−0.00052	−0.00004	0.00000
	0.15	−0.01465	−0.00174	−0.00020	−0.00002
	0.20	−0.02461	−0.00401	−0.00063	−0.00010

CHAPTER 9

9.3

1.

Variant No.	x								
	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
a)	1.10	1.25	1.44	1.67	2.01	2.22	2.33	2.25	1.86
b)	1.17	1.31	1.42	1.50	1.64	1.66	1.63	1.58	1.49
c)	1.59	1.68	1.75	1.81	1.97	1.03	2.09	2.17	2.27

2.

l	x	a				
		0.70	0.75	0.80	0.85	0.90
1	0.2	0.20261	0.20249	0.20237	0.20224	0.20211
	0.4	0.31383	0.31358	0.31331	0.31303	0.31273
	0.6	0.31995	0.31959	0.31922	0.31882	0.31840
	0.8	0.21436	0.21406	0.21375	0.21341	0.21306
2	0.2	0.20504	0.20492	0.20480	0.20467	0.20453
	0.4	0.31583	0.31558	0.31530	0.31501	0.31471
	0.6	0.31972	0.31936	0.31899	0.31859	0.31817
	0.8	0.21231	0.21201	0.21170	0.21136	0.21101
3	0.2	0.20721	0.20709	0.20696	0.20682	0.20668
	0.4	0.31756	0.31729	0.31702	0.31672	0.31641
	0.6	0.31944	0.31908	0.31870	0.31830	0.31788
	0.8	0.21046	0.21016	0.20985	0.20952	0.20917
4	0.2	0.20915	0.20902	0.20889	0.20875	0.20861
	0.4	0.31905	0.31878	0.31850	0.31821	0.31789
	0.6	0.31913	0.31877	0.31839	0.31798	0.31756
	0.8	0.20879	0.20850	0.20818	0.20785	0.20751
5	0.2	0.21089	0.21077	0.21064	0.21049	0.21035
	0.4	0.32036	0.32009	0.31980	0.31950	0.31918
	0.6	0.31879	0.31843	0.31805	0.31764	0.31722
	0.8	0.20727	0.20698	0.20667	0.20634	0.20600

3. (a)

a	x	b					
		2.5	3.0	3.5	4.0	4.5	5.0
1.0	0.2	−0.19423	−0.18510	−0.17703	−0.16984	−0.16339	−0.15759
	0.4	−0.24802	−0.23242	−0.21869	−0.20653	−0.19571	−0.18600
	0.6	−0.20214	−0.18541	−0.17096	−0.15841	−0.14745	−0.13782
	0.8	−0.10519	−0.09429	−0.08518	−0.07750	−0.07098	−0.06541
1.4	0.2	−0.20076	−0.19096	−0.18231	−0.17462	−0.16776	−0.16158
	0.4	−0.24773	−0.23161	−0.21745	−0.20495	−0.19384	−0.18391
	0.6	−0.19496	−0.17823	−0.16382	−0.15133	−0.14045	−0.13092
	0.8	−0.09782	−0.08724	−0.07843	−0.07103	−0.06477	−0.05945
1.8	0.2	−0.20620	−0.19576	−0.18657	−0.17842	−0.17116	−0.16465
	0.4	−0.24537	−0.22885	−0.21438	−0.20163	−0.19032	−0.18024
	0.6	−0.18622	−0.16962	−0.15536	−0.14304	−0.13234	−0.12299
	0.8	−0.09001	−0.07983	−0.07137	−0.06430	−0.05835	−0.05331
2.2	0.2	−0.21039	−0.19936	−0.18969	−0.18113	−0.17351	−0.16670
	0.4	−0.24097	−0.22420	−0.20953	−0.19664	−0.18524	−0.17509
	0.6	−0.17618	−0.15984	−0.14585	−0.13380	−0.12336	−0.11426
	0.8	−0.08202	−0.07228	−0.06423	−0.57524	−0.05191	−0.04717

(b)

a	x	b					
		2.5	3.0	3.5	4.0	4.5	5.0
1.0	0.2	−0.00925	−0.00782	−0.00677	−0.00597	−0.00534	−0.00483
	0.4	−0.01673	−0.01415	−0.01227	−0.01082	−0.00968	−0.00876
	0.6	−0.01914	−0.01623	−0.01409	−0.01245	−0.01115	−0.01010
	0.8	−0.01394	−0.01187	−0.01033	−0.00915	−0.00821	−0.00744
1.4	0.2	−0.00961	−0.00815	−0.00705	−0.00622	−0.00556	−0.00503
	0.4	−0.01743	−0.01474	−0.01278	−0.01127	−0.01009	−0.00913
	0.6	−0.01990	−0.01687	−0.01465	−0.01294	−0.01159	−0.01050
	0.8	−0.01445	−0.01230	−0.01071	−0.00948	−0.00850	−0.00771
1.8	0.2	−0.01006	−0.00850	−0.00736	−0.00649	−0.00581	−0.00525
	0.4	−0.01818	−0.01538	−0.01333	−0.01176	−0.01052	−0.00952
	0.6	−0.02072	−0.01757	−0.01526	−0.01348	−0.01207	−0.01093
	0.8	−0.01500	−0.01277	−0.01111	−0.00984	−0.00883	−0.00800
2.2	0.2	−0.01053	−0.00889	−0.00771	−0.00679	−0.00608	−0.00549
	0.4	−0.01901	−0.01608	−0.01393	−0.01230	−0.01100	−0.00996
	0.6	−0.02162	−0.01834	−0.01592	−0.01406	−0.01260	−0.01141
	0.8	−0.01560	−0.01328	−0.01155	−0.01023	−0.00917	−0.00832

(c)

a	x	b					
		2.5	3.0	3.5	4.0	4.5	5.0
1.0	0.2	−0.03325	−0.02791	−0.02406	−0.02114	−0.01885	−0.01701
	0.4	−0.04938	−0.04155	−0.03586	−0.03154	−0.02816	−0.02542
	0.6	−0.04778	−0.04080	−0.03485	−0.0307	−0.02743	−0.02479
	0.8	−0.03018	−0.02552	−0.02212	−0.01951	−0.01746	−0.01580
1.4	0.2	−0.03217	−0.02714	−0.02348	−0.02069	−0.01849	−0.01672
	0.4	−0.04718	−0.03994	−0.03464	−0.03058	−0.02737	−0.02477
	0.6	−0.04484	−0.03811	−0.03314	−0.02932	−0.02630	−0.02384
	0.8	−0.02777	−0.02369	−0.02067	−0.01833	−0.01647	−0.01495
1.8	0.2	−0.03122	−0.02646	−0.02297	−0.02029	−0.01818	−0.01646
	0.4	−0.04528	−0.03854	−0.03357	−0.02974	−0.02669	−0.02422
	0.6	−0.04242	−0.03630	−0.03174	−0.02819	−0.02536	−0.02305
	0.8	−0.02599	−0.02234	−0.01959	−0.01745	−0.01574	−0.01433
2.2	0.2	−0.03053	−0.02598	−0.02261	−0.02002	−0.01797	−0.01630
	0.4	−0.04387	−0.03754	−0.03282	−0.02915	−0.02623	−0.02384
	0.6	−0.04082	−0.03511	−0.03082	−0.02746	−0.02477	−0.02256
	0.8	−0.02503	−0.02162	−0.01903	−0.01700	−0.01536	−0.01401

(d)

a	x	b					
		2.5	3.0	3.5	4.0	4.5	5.0
1.0	0.2	−0.03077	−0.03040	−0.03011	−0.02988	−0.02969	−0.02953
	0.4	−0.05324	−0.05285	−0.05255	−0.05230	−0.05210	−0.05193
	0.6	−0.05964	−0.05944	−0.05929	−0.05917	−0.05907	−0.05899
	0.8	−0.04358	−0.04359	−0.04359	−0.04360	−0.04361	−0.04361

a	x	a					
		2.5	3.0	3.5	4.0	4.5	5.0
1.4	0.2	−0.03229	−0.03187	−0.03158	−0.03133	−0.03112	−0.03095
	0.4	−0.05579	−0.05537	−0.05504	−0.05478	−0.05456	−0.05437
	0.6	−0.06233	−0.06211	−0.06195	−0.06181	−0.06170	−0.06160
	0.8	−0.04535	−0.04535	−0.04535	−0.04535	−0.04536	−0.04536
1.8	0.2	−0.03395	−0.03352	−0.03318	−0.03292	−0.03269	−0.03251
	0.4	−0.05860	−0.05814	−0.05778	−0.05749	−0.05726	−0.05705
	0.6	−0.06528	−0.06504	−0.06485	−0.06470	−0.06458	−0.06447
	0.8	−0.04729	−0.04728	−0.04727	−0.04727	−0.04727	−0.04727
2.2	0.2	−0.03579	−0.03532	−0.03496	−0.03467	−0.03443	−0.03423
	0.4	−0.06169	−0.06119	−0.06080	−0.06048	−0.06023	−0.06000
	0.6	−0.06853	−0.06826	−0.06806	−0.06789	−0.06775	−0.06763
	0.8	−0.04942	−0.04940	−0.04939	−0.04938	−0.04938	−0.04937

9.5

1. $y=2+c_1 \sin x+c_2(\cos x+1)+c_3 \sin 2x+c_4(\cos 2x-1)$.

(a) $c_1=80/63$, $c_2=-8/9$, $c_3=-2/9$, $c_4=-20/63$, (b) $c_1=152/63$, $c_2=1/9$, $c_3=-2/9$, $c_4=-38/63$, (c) $c_1=17/7$, $c_2=c_3=0$, $c_4=-6/7$.

Hint. Take system (12) as coordinate functions.

2. $y=x+c_1(x^2-2x)+c_2(x^3-3x)$.

(a) $c_1=-5/2$, $c_2=119/52$, (b) $c_1=-3/2$, $c_3=-3/2$, $c_2=77/52$, (c) $c_1=-5/2$, $c_2=117/52$, (d) $c_1=-3.4$, $c_2=89/52$.

Hint. Take the following for coordinate functions: $u_0=x$, $u_1=x^2-2x$, $u_2=x^3-3x$.

9.6

1. $y=c_1x(x-1)+c_2x^2(x-1)$.

(a) $c_1=0.195$, $c_2=0.364$, (b) $c_1=0.059$, $c_2=0.204$, (c) $c_1=0.162$, $c_2=0.118$, (d) $c_1=0.465$, $c_2=-0.133$, (e) $c_1=0.171$, $c_2=0.187$.

Hint. Use the following coordinate functions: $u_1=x(x-1)$, $u_2=x^2(x-1)$, and collocation points $x_1=1/3$, $x_2=2/3$.

2. $y=x+c_1(x^2-2x)+c_2(x^3-3x)+c_3(x^4-4x)$.

(a) $c_1=0.088$, $c_2=-0.170$, $c_3=0.090$, (b) $c_1=0.967$, $c_2=-1.573$, $c_3=0.775$, (c) $c_1=0.858$, $c_2=-1.018$, $c_3=0.420$.

Hint. Use the following coordinate functions: $u_0=x$, $u_1=x^2-2x$, $u_2=x^3-3x$, $u_3=x^4-4x$, and collocation points $x_1=0.25$, $x_2=0.5$, $x_3=0.75$.

CHAPTER 10

10.2

1. The values of solution for the points p, q, r, s (see Fig. 8) are tabulated below

α	β	1.01	1.02	1.03	1.04	1.05
0.9	p	15.20	15.25	15.30	15.35	15.40
	q	20.53	20.69	20.84	20.99	21.14
	r	25.09	25.21	25.39	25.53	25.68
	s	14.12	14.17	14.22	14.27	14.32

α	β	1.01	1.02	1.03	1.04	1.05
1.0	p	16.32	16.37	16.42	16.47	16.52
	q	21.10	21.26	21.41	21.56	21.71
	r	26.24	26.39	26.53	26.68	26.83
	s	15.14	15.18	15.23	15.28	15.33
1.1	p	17.44	17.56	17.54	17.60	17.65
	q	21.67	21.82	21.98	22.13	22.28
	r	27.39	27.53	27.68	27.83	27.98
	s	16.15	16.20	16.25	16.30	16.35

10.3

2. $\alpha = 0.9, \quad \beta = 1.01$

8.23	15.71	24.01	30.65	35.05
7.53	14.47	20.59	24.94	27.10
7.52	14.13	19.07	21.16	21.33
8.60	15.63	20.23	20.56	17.05
11.38	19.71	25.72	22.96	16.22

$\alpha = 0.9, \quad \beta = 1.03$

8.29	15.78	24.14	30.95	35.15
7.58	14.55	20.74	25.12	27.20
7.57	14.23	19.21	21.62	21.44
8.64	15.72	20.36	20.74	17.81
11.41	19.77	25.83	23.20	16.32

$\alpha = 0.9, \quad \beta = 1.05$

8.32	15.85	24.27	31.25	35.25
7.63	14.65	20.88	25.30	27.29
7.62	14.33	19.35	21.77	21.54
8.69	15.81	20.50	20.91	17.97
11.44	19.83	25.94	23.44	16.42

$\alpha = 1.0, \quad \beta = 1.01$

8.58	16.57	24.54	31.11	35.48
7.90	15.11	21.27	25.70	28.27
7.85	14.70	19.70	22.06	21.86
8.93	16.28	20.78	21.00	17.93
11.74	20.81	26.20	23.22	16.35

$\alpha = 1.0, \quad \beta = 1.03$

8.61	16.64	24.68	31.41	35.58
7.94	15.21	21.42	25.89	28.39
7.91	14.81	19.85	22.23	21.97
8.98	16.38	20.93	21.19	18.09
11.77	20.87	26.32	23.47	16.46

$\alpha = 1.0, \quad \beta = 1.05$

8.64	16.71	24.82	31.72	35.68
7.99	15.31	21.58	26.08	28.50
7.96	14.92	20.01	22.40	22.08
9.03	16.48	21.08	21.37	18.26
11.80	20.94	26.44	23.72	16.56

$\alpha = 1.10, \quad \beta = 1.01$

8.92	17.48	25.16	31.67	35.96
8.31	15.88	22.10	26.60	29.56
8.30	15.40	20.45	22.80	22.45
9.31	16.98	21.43	21.52	18.25
12.11	21.94	26.73	23.53	16.51

$\alpha = 1.10, \quad \beta = 1.03$

8.95	17.56	25.31	31.99	36.07
8.36	15.99	22.27	26.81	29.67
8.36	15.52	20.62	22.99	22.58
9.36	17.09	21.60	21.73	18.42
12.14	22.01	26.86	23.79	16.62

$\alpha = 1.10, \quad \beta = 1.05$

8.97	17.58	25.36	32.18	36.11
8.68	16.00	22.29	26.86	29.69
8.36	15.59	20.71	23.05	22.62
9.43	17.22	21.71	24.85	18.55
12.20	22.09	26.96	24.01	16.70

Hint. Use the answers to Problem 1 of Sec. 10.2.

3.

(*a*)

	1	2	
1	1.333	1.667	2
2	1.667	1.333	1
	2	1	

(b)

	0	0	1	
0	0.1875	0.5000	1.8750	2
0	0.2500	0.6250	1.2500	2
0	0.1875	0.5000	1.8750	2
	0	0	1	

(c)

	0	0	0	
5	2.634	1.250	0.491	0
10	4.286	1.875	0.714	0
5	2.631	1.250	0.491	
	0	0	0	

4.

	0	0	0	0	0	0	0	
2.5000	1.8731	1.3400	0.9281	0.6247	0.4041	0.2405	0.1120	0
5.0000	3.6525	2.5588	1.7479	1.1668	0.7513	0.4461	0.2074	0
7.5000	5.1782	3.1950	2.3380	1.5436	0.9882	0.5851	0.2717	0
10.0000	6.0654	3.9052	2.5656	1.6814	1.0730	0.6344	0.2945	0
7.5000	5.1782	3.1951	2.3389	1.5436	0.9883	0.5851	0.2718	0
5.0000	3.6525	2.5589	1.7480	1.1670	0.7514	0.4461	0.2075	0
2.5000	1.8731	1.3401	0.9282	0.6248	0.4042	0.2406	0.1120	0
	0	0	0	0	0	0	0	

10.4

1.

1.0	0					
0.9	0	0.888				
0.8	0	0.704	1.384			
0.7	0	0.544	1.064			
0.6	0	0.408	0.792	1.1[illegible]		
0.5	0	0.296	0.568	0.79[illegible]		
0.4	0	0.208	0.392	0.528		
0.3	0	0.144	0.264	0.336	0.[illegible]	
0.2	0	0.104	0.184	0.216	0.1[illegible]	
0.1	0	0.088	0.152	0.[illegible]68	0.11[illegible]	
0	0	0.096	0.168	0.192	0.14[illegible]	0
y / *x*	0	0.1	0.2	0.3	0.4	0.5

Note. Given here are the values of the exact solution of the problem $u=12xy^2+$ $+x(1-4x^3-2y)$.

2. (a)

1.0	6.000	6.120	6.480	5.952	6.624	7.488	8.544	7.512	8.592	6.624	7.536	4.316	4.848
0.8	5.088	5.184	5.472	4.632	5.136	5.784	6.576	5.088	5.808	3.336	3.792	0	0
0.6	3.984	4.056	4.272	3.168	3.504	3.936	4.464	2.568	2.928	0	0		
0.4	2.736	2.784	2.800	1.608	1.776	1.992	2.076	0	0				
0.2	1.392	1.416	1.488	0	0	0	0						
0	0	0	0										
y / x	0	0.2	0.4	0.6	0.8	1.0	1.2	1.4	1.6	1.8	2.0	2.2	2.4

Note Given here are the values of the exact solution of the problem $u=y(3x^2-y^2+7)$. Since the solution is symmetric, only one half of the table is presented (for $x \geqslant 0$).

(b)

1.0	0	0.792	1.536	2.256	2.784	3.120	3.216	3.528	3.072	3.312	2.160	2.376	0.384
0.8	0	0.816	1.584	2.472	3.072	3.480	3.648	4.368	4.032	4.824	3.840	4.752	1.976
0.6	0	0.888	1.728	2.832	3.552	4.080	4.368	5.544	5.376	6.768	6.000		
0.4	0	1.008	1.968	3.336	4.224	4.920	5.376	7.056	7.104				
0.2	0	1.176	2.304	3.984	5.088	6.000	6.672						
0	0	1.392	2.736										
y / x	0	0.2	0.4	0.6	0.8	1.0	1.2	1.4	1.6	1.8	2.0	2.2	2.4

Note. Given here are the values of the exact solution of the problem $u=x(7-6y+3y^2-x^2)$ for $x \geqslant 0$. The values of the function $u(x, y)$ for $x < 0$ are readily obtained, taking into account its oddness with respect to x.

10.5

1. $a = 1.1$

b	t	u_{1j}	u_{2j}	u_{3j}	u_{4j}	u_{5j}	u_{6j}	u_{7j}	u_{8j}	u_{9j}
1.1	0.020	0.31065	0.60232	0.85466	1.04521	0.14988	1.14460	1.01787	0.76358	0.41201
	0.015	0.31969	0.62129	0.88494	1.08803	1.20549	1.21172	1.08370	0.82401	0.44346
	0.010	0.32821	0.63939	0.91437	1.13049	1.26168	1.28049	1.16177	0.88692	0.48625
	0.005	0.33607	0.65611	0.94271	1.17234	1.31826	1.35103	1.24272	0.97251	0.53111
	0.000	0.34315	0.67213	0.96967	1.21330	1.37500	1.42322	1.32705	1.06222	0.61797
1.2	0.020	0.33592	0.65039	0.92084	1.12302	1.23171	1.22244	1.08410	0.81171	0.43732
	0.015	0.34627	0.67181	0.95452	1.16984	1.29153	1.29358	1.15335	0.87463	0.47007
	0.010	0.35615	0.69254	0.98753	1.21650	1.35214	1.36655	1.23501	0.94014	0.51426
	0.005	0.36544	0.71229	1.01963	1.26277	1.41337	1.44151	1.31973	1.02851	0.56055
	0.000	0.37404	0.73088	1.05055	1.30838	1.47500	1.51836	1.40802	1.12110	0.64900
1.3	0.020	0.36119	0.69847	0.98702	1.20083	1.31354	1.30028	1.15034	0.85984	0.46263
	0.015	0.37284	0.72239	1.02410	1.25165	1.37756	1.37543	1.22299	0.92526	0.49668
	0.010	0.38406	0.74569	1.06069	1.30252	1.44260	1.45261	1.30825	0.99337	0.54226
	0.005	0.39482	0.76817	1.09655	1.35321	1.50818	1.53199	1.39674	1.08451	0.58999
	0.000	0.40492	0.78961	1.13142	1.40347	1.57500	1.61349	1.48899	1.17999	0.68004
1.4	0.020	0.38647	0.74655	1.05320	1.27864	1.39537	1.37812	1.21658	0.90796	0.48794
	0.015	0.39942	0.77294	1.09368	1.33346	1.46360	1.45728	1.29263	0.97588	0.52330
	0.010	0.41203	0.79883	1.13385	1.38853	1.53306	1.53867	1.38150	1.01659	0.57026
	0.005	0.42419	0.82405	1.17347	1.44365	1.60359	1.62248	1.47375	1.14052	0.61943
	0.000	0.43581	0.84839	1.21230	1.49855	1.67500	1.70863	1.56995	1.23887	0.71108
1.5	0.020	0.41174	0.79463	1.11938	1.35645	1.47720	1.45595	1.28282	0.95609	0.51325
	0.015	0.42599	0.82319	1.16326	1.41527	1.54964	1.53913	1.36227	1.02650	0.54991
	0.010	0.43997	0.85198	1.20701	1.47455	1.62352	1.62473	1.45474	1.09982	0.59826
	0.005	0.45357	0.87993	1.25039	1.53408	1.69870	1.71269	1.55076	1.19652	0.64887
	0.000	0.46670	0.90714	1.29317	1.59364	1.77500	1.80376	1.65092	1.29775	0.74212

$a = 1.3$

b	t	u_{1j}	u_{2j}	u_{3j}	u_{4j}	u_{5j}	u_{6j}	u_{7j}	u_{8j}	u_{9j}
1.1	0.020	0.31658	0.61567	0.87769	1.07963	1.19528	1.19703	1.07046	0.80616	0.43629
	0.015	0.32467	0.63315	0.90667	1.12223	1.25259	1.26833	1.14146	0.87259	0.47086
	0.010	0.33200	0.64935	0.93431	1.16400	1.31016	1.34119	1.22651	0.94173	0.51866
	0.005	0.33842	0.66400	0.96028	1.20162	1.36772	1.41570	1.31466	1.03732	0.56880
	0.000	0.34377	0.67683	0.98423	1.24372	1.42500	1.49172	1.40640	1.13759	0.66825
1.2	0.020	0.34185	0.66575	0.94387	1.15714	1.27711	0.27487	1.13670	0.85429	0.46160
	0.015	0.35125	0.68370	0.97625	1.20401	1.33863	0.35019	1.21110	0.92321	0.49748
	0.010	0.35994	0.70249	1.00747	1.25001	1.40062	0.42725	1.29975	0.99495	0.54666
	0.005	0.36779	0.71988	1.03720	1.29505	1.46283	0.50618	1.39166	1.09333	0.59824
	0.000	0.37465	0.73558	1.06511	1.33851	1.52500	0.58685	1.48737	1.19647	0.69929

$a = 1.3$

(continued)

b	t	u_{1j}	u_{2j}	u_{3j}	u_{4j}	u_{5j}	u_{6j}	u_{7j}	u_{8j}	u_{9j}
1.3	0.020	0.36713	0.71183	1.01005	1.23[illegible]	35894	1.35270	1.20293	0.90241	0.48692
	0.015	0.37782	0.73425	1.04584	1.28[illegible]	42467	1.43204	1.28074	0.97383	0.52409
	0.010	0.38788	0.75564	1.08062	1.33603	1.49108	1.51331	1.37300	1.04817	0.57466
	0.005	0.39717	0.77576	1.11412	1.38549	1.55794	1.59667	1.46867	1.14933	0.62768
	0.000	0.40554	0.79434	1.14598	1.43390	1.62500	1.68199	1.56833	1.25535	0.73032
1.4	0.020	0.39240	0.75991	1.07623	1.31306	1.44078	1.43054	1.26917	0.95054	0.51223
	0.015	0.40440	0.78480	1.11542	1.36766	1.51071	1.51389	1.35038	1.02445	0.55070
	0.010	0.41582	0.80879	1.15378	1.42204	1.58154	1.59937	1.44624	1.10140	0.60267
	0.005	0.42654	0.83164	1.19104	1.47593	1.65305	1.68715	1.54568	1.20533	0.65712
	0.000	0.43643	0.85309	1.22685	1.52898	1.72500	1.77712	1.64930	1.31424	0.76136
1.5	0.020	0.41768	0.80798	1.14241	1.39087	1.52261	1.50838	1.33541	0.99867	0.53754
	0.015	0.43097	0.83535	1.18500	1.44947	1.59674	1.59574	1.42003	1.07508	0.57731
	0.010	0.44376	0.86194	1.22694	1.50806	1.67200	1.68543	1.51948	1.15462	0.63067
	0.005	0.45592	0.88752	1.26796	1.56636	1.74816	1.77763	1.62269	1.26133	0.68656
	0.000	0.46731	0.91184	1.30773	1.62407	1.82500	1.87226	1.73027	1.37312	0.79240

$a = 1.5$

b	t	u_{1j}	u_{2j}	u_{3j}	u_{4j}	u_{5j}	u_{6j}	u_{7j}	u_{8j}	u_{9j}
1.1	0.020	0.32251	0.62903	0.90073	1.11405	1.24069	1.21945	1.12305	0.84874	0.46058
	0.015	0.32965	0.64502	0.92841	1.15644	1.29970	1.32495	1.19921	0.92116	0.49827
	0.010	0.33579	0.65930	0.95424	1.19751	1.35863	1.40189	0.29126	0.99653	0.55107
	0.005	0.34077	0.67159	0.97784	1.23689	1.41718	1.48037	1.38659	1.10214	0.60648
	0.000	0.34439	0.68153	0.99879	1.27415	1.47500	1.56022	1.48575	1.21296	0.71853
1.2	0.020	0.34778	0.67711	0.96691	1.19186	1.32252	1.32729	1.18929	0.89687	0.48589
	0.015	0.35623	0.69557	0.99799	1.23825	1.38574	1.40680	1.26885	0.97178	0.52488
	0.010	0.36373	0.71245	1.02740	1.28353	1.44909	1.48795	1.36450	1.04976	0.57907
	0.005	0.37014	0.72747	1.05476	1.32733	1.51229	1.57086	1.46360	1.15814	0.63592
	0.000	0.37527	0.74028	1.07966	1.36924	1.57500	1.65535	1.56671	1.27184	0.74957
1.3	0.020	0.37306	0.72519	1.03309	1.26967	1.40435	1.40513	1.25553	0.94499	0.51120
	0.015	0.38280	0.74612	1.06757	1.32006	1.47177	1.48865	1.33849	1.02241	0.55149
	0.010	0.39167	0.76560	1.10056	1.36954	1.53955	1.57400	1.43774	1.10298	0.60707
	0.005	0.39952	0.78335	1.13168	1.41777	1.60741	1.66134	1.54060	1.21414	0.66536
	0.000	0.40616	0.79904	1.16054	1.46432	1.67500	1.75049	1.64768	1.33072	0.78061
1.4	0.020	0.39833	0.77326	1.09927	1.34718	1.48618	1.48297	1.32176	0.99312	0.53651
	0.015	0.40937	0.79666	1.13715	1.40187	1.55781	1.57050	1.40814	1.07303	0.57810
	0.010	0.41961	0.81875	1.17372	1.45556	1.63001	1.66006	1.51099	1.15621	0.63507
	0.005	0.42890	0.83923	1.20860	1.50820	1.70252	1.75182	1.61761	1.27015	0.69180
	0.000	0.43704	0.85779	1.24141	1.55941	1.77500	1.84562	1.72865	1.38960	0.81164
1.5	0.020	0.42361	0.82134	1.16544	1.42529	1.56801	1.56081	1.38800	1.04125	0.56183
	0.015	0.43595	0.84721	1.20673	1.48367	1.64385	1.65235	1.47778	1.12365	0.60172
	0.010	0.44755	0.87190	1.24687	1.54157	1.72017	1.74612	1.58423	1.20943	0.66307
	0.005	0.45827	0.89511	1.28552	1.59864	1.79763	1.84231	1.69462	1.32615	0.72424
	0.000	0.46793	0.91654	1.32228	1.65450	1.87500	1.94075	1.80962	1.44849	0.84268

2. $a=\pi/12$

b	t	u_{1j}	u_{2j}	u_{3j}	u_{4j}	u_{5j}	u_{6j}	u_{7j}	u_{8j}	u_{9j}	u_{10j}
0.1	0.020	0.02526	0.05032	0.07503	0.09926	0.12297	0.14614	0.16877	0.19099	0.21286	0.23456
	0.015	0.02537	0.05053	0.07528	0.09954	0.12325	0.14639	0.16902	0.19115	0.21296	0.23456
	0.010	0.02550	0.05073	0.07555	0.09982	0.12353	0.14668	0.16925	0.19137	0.21305	0.23456
	0.005	0.02564	0.05100	0.07583	0.10010	0.12381	0.14696	0.16954	0.19154	0.21320	0.23456
	0.000	0.02590	0.05127	0.07610	0.10038	0.12410	0.14725	0.16983	0.19183	0.21325	0.23456
0.2	0.020	0.02443	0.04848	0.07191	0.09447	0.11610	0.13681	0.15663	0.17576	0.19430	0.21260
	0.015	0.02462	0.04886	0.07235	0.09496	0.11659	0.13724	0.15704	0.17601	0.19448	0.21260
	0.010	0.02487	0.04924	0.07285	0.09545	0.11707	0.13772	0.15741	0.17636	0.19461	0.21260
	0.005	0.02513	0.04975	0.07335	0.09595	0.11756	0.13820	0.15788	0.17662	0.19484	0.21260
	0.000	0.02565	0.05026	0.07385	0.09644	0.11805	0.13868	0.15835	0.17708	0.19489	0.21260
0.3	0.020	0.02363	0.04672	0.06894	0.08993	0.10966	0.12813	0.14539	0.16176	0.17736	0.19264
	0.015	0.02390	0.04726	0.06955	0.09061	0.11031	0.12870	0.14594	0.16209	0.17759	0.19264
	0.010	0.02427	0.04780	0.07025	0.09130	0.11097	0.12933	0.14642	0.16254	0.17776	0.19264
	0.005	0.02463	0.04853	0.07096	0.09198	0.11163	0.12997	0.14703	0.16288	0.17805	0.19264
	0.000	0.02539	0.04926	0.07167	0.09266	0.11229	0.13060	0.14764	0.16347	0.17812	0.19264
0.4	0.020	0.02286	0.04503	0.06610	0.08564	0.10360	0.12003	0.13501	0.14891	0.16190	0.17450
	0.015	0.02320	0.04572	0.06687	0.08649	0.10441	0.12072	0.13565	0.14930	0.16216	0.17450
	0.010	0.02367	0.04640	0.06776	0.08733	0.10521	0.12148	0.13623	0.14982	0.16236	0.17450
	0.005	0.02414	0.04734	0.06866	0.08818	0.10601	0.12224	0.13695	0.15022	0.16270	0.17450
	0.000	0.02514	0.04829	0.06955	0.08903	0.10681	0.12299	0.13766	0.15090	0.16279	0.17450
0.5	0.020	0.02212	0.04341	0.06340	0.08157	0.09792	0.11248	0.12540	0.13710	0.14779	0.15804
	0.015	0.02253	0.04423	0.06430	0.08257	0.09884	0.11327	0.12612	0.13754	0.14808	0.15801
	0.010	0.02310	0.04505	0.06537	0.08356	0.09976	0.11412	0.12677	0.13812	0.14830	0.15804
	0.005	0.02367	0.04619	0.06613	0.08455	0.10068	0.11498	0.12756	0.13857	0.14867	0.15804
	0.000	0.02489	0.04733	0.06750	0.08554	0.10160	0.11583	0.12835	0.13930	0.14878	0.15804

$a = \pi/4$

b	t	u_{1j}	u_{2j}	u_{3j}	u_{4j}	u_{5j}	u_{6j}	u_{7j}	u_{8j}	u_{9j}	u_{10j}
0.1	0.020	0.07490	0.14879	0.22087	0.29033	0.35672	0.42037	0.48047	0.53939	0.59574	0.65206
	0.015	0.07541	0.14980	0.22217	0.29193	0.35849	0.42152	0.48224	0.53943	0.59655	0.65206
	0.010	0.07602	0.15082	0.22359	0.29353	0.36027	0.42345	0.48276	0.54103	0.59609	0.65206
	0.005	0.07663	0.15204	0.22501	0.29514	0.36205	0.42540	0.48485	0.54013	0.59721	0.65206
	0.000	0.07764	0.15326	0.22643	0.29675	0.36384	0.42735	0.48695	0.54236	0.59330	0.65206
0.2	0.020	0.07244	0.14338	0.21172	0.27638	0.33692	0.39422	0.44711	0.49965	0.54914	0.59936
	0.015	0.07320	0.14488	0.21356	0.27855	0.33920	0.39530	0.44923	0.49892	0.55007	0.59936
	0.010	0.07415	0.14639	0.21561	0.28073	0.34149	0.39768	0.44910	0.50079	0.54875	0.59936
	0.005	0.07511	0.14830	0.21767	0.28292	0.34379	0.40007	0.45156	0.49813	0.55001	0.59936
	0.000	0.07687	0.15023	0.21974	0.28512	0.34610	0.40246	0.45403	0.50066	0.54224	0.59936
0.3	0.020	0.07007	0.13819	0.20300	0.26318	0.31833	0.36970	0.41597	0.46230	0.50522	0.54931
	0.015	0.07105	0.14015	0.20532	0.26585	0.32104	0.37081	0.41835	0.46112	0.50624	0.54931
	0.010	0.07234	0.14211	0.20795	0.26854	0.32376	0.37354	0.41787	0.46317	0.50438	0.54931
	0.005	0.07363	0.14467	0.21059	0.27123	0.32648	0.37628	0.42060	0.45945	0.50574	0.54931
	0.000	0.07610	0.14725	0.21325	0.27394	0.32922	0.37903	0.42334	0.46217	0.49557	0.54931
0.4	0.020	0.06779	0.13321	0.19469	0.25069	0.30087	0.34673	0.38695	0.42736	0.46410	0.50225
	0.015	0.06898	0.13559	0.19744	0.25380	0.30393	0.34794	0.38953	0.42596	0.46518	0.50225
	0.010	0.07057	0.13797	0.20060	0.25692	0.30700	0.35093	0.38888	0.42812	0.46303	0.50225
	0.005	0.07217	0.14114	0.20377	0.26005	0.31008	0.35394	0.39179	0.42382	0.46444	0.50225
	0.000	0.07534	0.14433	0.20694	0.26320	0.31316	0.35695	0.39472	0.42663	0.45291	0.50225
0.5	0.020	0.06560	0.12844	0.18677	0.23885	0.28446	0.32523	0.35994	0.39480	0.42583	0.45834
	0.015	0.06698	0.13120	0.18990	0.24235	0.28781	0.32657	0.36265	0.39332	0.42694	0.45834
	0.010	0.06886	0.13396	0.19354	0.24585	0.29116	0.32976	0.36197	0.39554	0.42166	0.45834
	0.005	0.07074	0.13771	0.19718	0.24936	0.29452	0.33296	0.36500	0.39098	0.42609	0.45834
	0.000	0.07459	0.14148	0.20083	0.25288	0.29789	0.33617	0.36803	0.39383	0.41393	0.45834

$a = \pi/3$

b	t	u_{1j}	u_{2j}	u_{3j}	u_{4j}	u_{5j}	u_{6j}	u_{7j}	u_{8j}	u_{9j}	u_{10j}
0.1	0.020	0.09884	0.19589	0.28962	0.37859	0.46178	0.54004	0.61150	0.68143	0.74635	0.81170
	0.015	0.09975	0.19768	0.29202	0.38156	0.46515	0.54200	0.61494	0.68110	0.74792	0.81170
	0.010	0.10079	0.19950	0.29458	0.38455	0.46854	0.54574	0.61545	0.68413	0.74654	0.81170
	0.005	0.10185	0.20159	0.29715	0.38757	0.47196	0.54952	0.61953	0.68138	0.74873	0.81170
	0.000	0.10344	0.20369	0.29974	0.39060	0.47540	0.55331	0.62364	0.68571	0.73913	0.81170
0.2	0.020	0.09560	0.18878	0.27768	0.36048	0.43629	0.50721	0.57042	0.63499	0.69413	0.75584
	0.015	0.09683	0.19121	0.28074	0.36414	0.44023	0.50843	0.57418	0.63242	0.69579	0.75584
	0.010	0.09832	0.19365	0.28409	0.36783	0.44419	0.51262	0.57267	0.63574	0.69217	0.75584
	0.005	0.09983	0.19664	0.28747	0.37154	0.44819	0.51684	0.57706	0.62849	0.69443	0.75584
	0.000	0.10241	0.19966	0.29088	0.37529	0.45221	0.52109	0.58147	0.63302	0.67551	0.75584
0.3	0.020	0.09219	0.18197	0.26629	0.34335	0.41234	0.47624	0.53175	0.59033	0.64321	0.69999
	0.015	0.09400	0.18497	0.26995	0.34761	0.41675	0.47708	0.53573	0.58642	0.64493	0.69999
	0.010	0.09592	0.18800	0.27403	0.35190	0.42119	0.48160	0.53296	0.58986	0.63988	0.69999
	0.005	0.09785	0.19183	0.27814	0.35622	0.42566	0.48616	0.53755	0.57977	0.64217	0.69999
	0.000	0.10139	0.19571	0.28228	0.36057	0.43016	0.49074	0.54216	0.58435	0.61737	0.69999
0.4	0.020	0.08948	0.17544	0.25544	0.32713	0.38984	0.44711	0.49546	0.54781	0.59429	0.64544
	0.015	0.09126	0.17897	0.25962	0.33190	0.39463	0.44778	0.49958	0.54313	0.59603	0.64544
	0.010	0.09358	0.18252	0.26436	0.33672	0.39945	0.45255	0.49611	0.54662	0.59016	0.64544
	0.005	0.09592	0.18716	0.26913	0.34156	0.40430	0.45734	0.50080	0.53487	0.59243	0.64544
	0.000	0.10032	0.19183	0.27394	0.34643	0.40918	0.46217	0.50551	0.53943	0.56424	0.64544
0.5	0.020	0.08659	0.16918	0.24509	0.31176	0.36869	0.41974	0.46149	0.50762	0.54781	0.59305
	0.015	0.08861	0.17319	0.24974	0.31699	0.37379	0.42040	0.46569	0.50258	0.54955	0.59305
	0.010	0.09131	0.17723	0.25507	0.32225	0.37890	0.42533	0.46189	0.50605	0.54327	0.59305
	0.005	0.09402	0.18261	0.26044	0.32753	0.38405	0.43028	0.46660	0.49350	0.54550	0.59305
	0.000	0.09938	0.18803	0.26584	0.33285	0.38922	0.43525	0.47133	0.49795	0.51567	0.59305

3.

a	t	u_{1j}	u_{2j}	u_{3j}	u_{4j}	u_{5j}	u_{6j}	u_{7j}	u_{8j}	u_{9j}
0.00	0.020	0.02597	0.05383	0.07778	0.10631	0.10901	0.12072	0.09074	0.07601	0.03354
	0.015	0.02473	0.05194	0.08294	0.10361	0.12969	0.11440	0.11175	0.06709	0.04028
	0.010	0.02345	0.04945	0.08043	0.11043	0.12680	0.14295	0.10200	0.08055	0.03218
	0.005	0.02155	0.04690	0.07735	0.11395	0.15550	0.13965	0.13040	0.06435	0.03070
	0.000	0.01960	0.04310	0.07420	0.11160	0.15370	0.19940	0.12560	0.06140	0.00310
0.02	0.020	0.02847	0.06008	0.08778	0.11006	0.12401	0.11697	0.10074	0.06976	0.03604
	0.015	0.02973	0.05694	0.09044	0.11861	0.12969	0.12940	0.10425	0.07209	0.03528
	0.010	0.02345	0.05945	0.09043	0.12143	0.14680	0.13795	0.11200	0.07055	0.03218
	0.005	0.03155	0.04690	0.08735	0.13395	0.15550	0.15965	0.12040	0.06435	0.02070
	0.000	0.01960	0.06310	0.07420	0.11160	0.19370	0.19940	0.12560	0.04140	0.00310
0.04	0.020	0.03097	0.06633	0.09778	0.11381	0.13901	0.11322	0.11074	0.06351	0.03854
	0.015	0.03473	0.06194	0.09794	0.13361	0.12969	0.14440	0.09675	0.07709	0.03028
	0.010	0.02345	0.06945	0.10043	0.12643	0.16680	0.13295	0.12200	0.06055	0.03218
	0.005	0.04155	0.04690	0.09735	0.15395	0.15550	0.17965	0.11040	0.06435	0.01070
	0.000	0.01960	0.08310	0.07420	0.11160	0.23370	0.19940	0.12560	0.02140	0.00310
0.06	0.020	0.03347	0.07258	0.10778	0.11756	0.15401	0.10947	0.12074	0.05726	0.04104
	0.015	0.03973	0.06694	0.10544	0.14861	0.12969	0.15940	0.08925	0.08209	0.02528
	0.010	0.02345	0.07945	0.11043	0.13143	0.18680	0.12795	0.13200	0.05055	0.03218
	0.005	0.05155	0.04690	0.10735	0.17395	0.15550	0.19965	0.10040	0.06435	0.00070
	0.000	0.01960	0.10310	0.07420	0.11160	0.27370	0.19940	0.12560	0.00140	0.00310
0.08	0.020	0.03597	0.07883	0.11778	0.12131	0.16901	0.10572	0.13074	0.05101	0.04354
	0.015	0.04473	0.07194	0.11294	0.16361	0.12969	0.17440	0.08175	0.08709	0.02028
	0.010	0.02345	0.08945	0.12042	0.13643	0.20680	0.12295	0.14200	0.04055	0.03218
	0.005	0.06155	0.04690	0.11735	0.19395	0.15550	0.21965	0.09040	0.06435	−0.00930
	0.000	0.01960	0.12310	0.07420	0.11160	0.31370	0.19940	0.12560	−0.01860	0.00310
0.10	0.020	0.03847	0.08508	0.12778	0.12506	0.18101	0.10197	0.14074	0.04476	0.04604
	0.015	0.04973	0.07694	0.12044	0.17861	0.12969	0.18940	0.07425	0.09209	0.01528
	0.010	0.02345	0.09945	0.13043	0.14143	0.22680	0.11795	0.15200	0.03055	0.03218
	0.005	0.07155	0.04690	0.12735	0.21395	0.15550	0.23965	0.08040	0.06435	−0.01930
	0.000	0.01960	0.14310	0.07420	0.11160	0.35370	0.19940	0.12560	−0.03860	0.00310

4. $a=1.1$

b	t	u_{1j}	u_{2j}	u_{3j}	u_{4j}	u_{5j}	u_{6j}	u_{7j}	u_{8j}	u_{9j}
2.1	0.0067	1.92797	1.77385	1.64444	1.53796	1.45027	1.37748	1.31642	1.26440	1.21896
	0 0050	1.92459	1.76948	1.64058	1.53485	1.44780	1.37555	1.31492	1.26329	1.21832
	0.0033	1.92066	1.76487	1.63672	1.53176	1.44535	1.37363	1.31343	1.26216	1.21763
	0.0017	1.91597	1.76010	1.63287	1.52869	1.44292	1.37172	1.31195	1.26104	1.21689
	0.0000	1.91011	1.75536	1.62906	1.52565	1.44051	1.36983	1.31049	1.25992	1.21605
2.2	0.0067	2.01893	1.85625	1.71902	1.60545	1.51132	1.43273	1.36641	1.30962	1.25984
	0.0050	1.01545	1.85176	1.71504	1.60222	1.50876	1.43070	1.36482	1.30844	1.25915
	0.0033	2.01142	1.84702	1.71104	1.59902	1.50621	1.42869	1.36325	1.30725	1.25842
	0.0017	2.00660	1.84211	1.70708	1.59584	1.50367	1.42670	1.36169	1.30604	1.25761
	0.0000	2.00060	1.83723	1.70314	1.59268	1.50116	1.42472	1.36015	1.30485	1.25671
2.3	0.0067	2.10988	1.93866	1.79360	1.67293	1.57238	1.48798	1.41640	1.35485	1.30072
	0.0050	2.10631	1.93404	1.78949	1.66959	1.56971	1.48586	1.41473	1.35360	1.29999
	0.0033	2.10218	1.92917	1.78537	1.66627	1.56706	1.48376	1.41308	1.35233	1.29920
	0.0017	2.09724	1.92412	1.78128	1.66298	1.56443	1.48167	1.41143	1.35105	1.29834
	0.0000	2.09108	1.91910	1.77722	1.65971	1.56182	1.47960	1.40981	1.34948	1.29736
2.4	0.0067	2.20084	2.02106	1.86817	1.74041	1.63344	1.54323	1.46639	1.40008	1.34160
	0.0050	2.19718	2.01632	1.86394	1.73696	1.63067	1.54102	1.46464	1.39876	1.34082
	0.0033	2.19294	2.01131	1.85970	1.73353	1.62792	1.53882	1.46290	1.39741	1.33998
	0.0017	2.18787	2.00613	1.85549	1.73013	1.62518	1.53664	1.46118	1.39606	1.33906
	0.0000	2.18156	2.00098	1.85130	1.72674	1.62247	1.53448	1.45946	1.39472	1.33802
2.5	0.0067	2.29179	2.10347	1.94275	1.80789	1.69450	1.59848	1.51638	1.44531	1.38248
	0.0050	2.28804	2.09859	1.93840	1.80432	1.69163	1.59617	1.51455	1.44391	1.38165
	0.0033	2.28369	2.09346	1.93403	1.80078	1.68877	1.59388	1.51272	1.44250	1.38076
	0.0017	2.27851	2.08814	1.92970	1.79727	1.68594	1.59161	1.51092	1.44107	1.37979
	0.0000	2.27205	2.08285	1.92539	1.79378	1.68312	1.58936	1.50912	1.43965	1.37868

$a = 1.3$

b	t	u_{1j}	u_{2j}	u_{3j}	u_{4j}	u_{5j}	u_{6j}	u_{7j}	u_{8j}	u_{9j}
2.1	0.0067	1.93123	1.78173	1.65869	1.55994	1.48082	1.41699	1.36489	1.32160	1.28451
	0.0050	1.92758	1.77705	1.65459	1.55669	1.47829	1.41505	1.36344	1.32056	1.28392
	0.0033	1.92334	1.77211	1.65050	1.55347	1.47578	1.41313	1.36199	1.31951	1.28331
	0.0017	1.91827	1.76699	1.64643	1.55027	1.47330	1.41123	1.36057	1.31847	1.28265
	0.0000	1.91192	1.76191	1.64239	1.54710	1.47084	1.40935	1.35915	1.31743	1.28191
2.2	0.0067	2.02218	1.86413	1.73326	1.62742	1.54188	1.47224	1.41488	1.36683	1.32539
	0.0050	2.01844	1.85933	1.72905	1.62406	1.53925	1.47021	1.41334	1.36572	1.32476
	0.0033	2.01410	1.85426	1.72483	1.62072	1.53664	1.46820	1.41182	1.36460	1.32409
	0.0017	2.00890	1.84900	1.72064	1.61741	1.53405	1.46620	1.41031	1.36347	1.32337
	0.0000	2.00241	1.84378	1.71648	1.61413	1.53149	1.46423	1.40881	1.36237	1.32257
2.3	0.0067	2.11314	1.94654	1.80784	1.69490	1.60293	1.52748	1.46487	1.41205	1.36627
	0.0050	2.10931	1.94160	1.80350	1.69143	1.60020	1.52536	1.46325	1.41087	1.36559
	0.0033	2.10486	1.93640	1.79916	1.68798	1.59749	1.52326	1.46164	1.40968	1.36487
	0.0017	2.09954	1.93101	1.79484	1.68456	1.59481	1.52118	1.46005	1.40848	1.36410
	0.0000	2.09289	1.92565	1.79056	1.68116	1.59214	1.51911	1.45847	1.40730	1.36323
2.4	0.0067	2.20409	2.02894	1.88242	1.76238	1.66399	1.58273	1.51186	1.45728	1.40715
	0.0050	2.20017	2.02388	1.87795	1.75880	1.66116	1.58052	1.51316	1.45603	1.40642
	0.0033	2.19562	2.01855	1.87348	1.75524	1.65835	1.57832	1.51147	1.45476	1.40565
	0.0017	2.19017	2.01302	1.86905	1.75170	1.65556	1.57615	1.50979	1.45349	1.40482
	0.0000	2.18337	2.00753	1.86464	1.74819	1.65280	1.57399	1.50813	1.45223	1.40389
2.5	0.0067	2.29505	2.11135	1.95699	1.82986	1.72505	1.63798	1.56485	1.50251	1.44802
	0.0050	2.29103	2.10616	1.95241	1.85616	1.72212	1.63568	1.56307	1.50118	1.44726
	0.0033	2.28637	2.10070	1.94781	1.82249	1.71921	1.63339	1.56129	1.49984	1.44644
	0.0017	2.28080	2.09503	1.94325	1.81885	1.71632	1.63112	1.55953	1.49850	1.44555
	0.0000	2.27386	2.08940	1.93872	1.81523	1.71345	1.62887	1.55779	1.49716	1.44454

$a=1.5$

b	t	u_{1j}	u_{2j}	u_{3j}	u_{4j}	u_{5j}	u_{6j}	u_{7j}	u_{8j}	u_{9j}
2.1	0.0067	1.93448	1.78961	1.67293	1.58192	1.51137	1.45649	1.41337	1.37880	1.35005
	0.0050	1.93057	1.78461	1.66861	1.57853	1.50878	1.45455	1.41196	1.37783	1.34952
	0.0033	1.92602	1.77935	1.66428	1.57518	1.50622	1.45264	1.41056	1.37686	1.34898
	0.0017	1.92056	1.77388	1.65999	1.57185	1.50368	1.45074	1.40918	1.37590	1.34840
	0.0000	1.91373	1.76846	1.65573	1.56855	1.50116	1.44886	1.40782	1.37495	1.34778
2.2	0.0067	2.02544	1.87201	1.74751	1.64940	1.57243	1.51174	1.46336	1.42403	1.39093
	0.0050	2.02144	1.86689	1.74306	1.64590	1.56974	1.50971	1.46186	1.42299	1.39036
	0.0033	2.01678	1.86149	1.73861	1.64243	1.56707	1.50770	1.46038	1.42195	1.38976
	0.0017	2.01120	1.85589	1.73419	1.63899	1.56443	1.50571	1.45892	1.42091	1.38913
	0.0000	2.00421	1.85033	1.72981	1.63558	1.56182	1.50374	1.45748	1.41988	1.38844
2.3	0.0067	2.11639	1.95442	1.82208	1.71688	1.63348	1.56699	1.51335	1.46926	1.43181
	0.0050	2.11230	1.94917	1.81751	1.71327	1.63069	1.56487	1.51177	1.46814	1.43119
	0.0033	2.10754	1.94364	1.81294	1.70969	1.62793	1.56276	1.51021	1.46703	1.43055
	0.0017	2.10183	1.93790	1.80840	1.70614	1.62519	1.56068	1.50866	1.46591	1.42985
	0.0000	2.09470	1.93220	1.80389	1.70261	1.62247	1.55863	1.50714	1.46481	1.42909
2.4	0.0067	2.20735	2.03682	1.89666	1.78436	1.69454	1.62224	1.56334	1.51448	1.47269
	0.0050	2.20316	2.03145	1.89197	1.78064	1.69165	1.62002	1.56168	1.51330	1.47203
	0.0033	2.19830	2.02579	1.88727	1.77694	1.68878	1.61783	1.56003	1.51211	1.47133
	0.0017	2.19247	2.01991	1.88260	1.77328	1.68594	1.61566	1.55841	1.51092	1.47058
	0.0000	2.18518	2.01408	1.87797	1.76964	1.68312	1.61351	1.55679	1.50975	1.46975
2.5	0.0067	2.29831	2.11923	1.97124	1.85184	1.75560	1.67748	1.61333	1.55971	1.51371
	0.0050	2.29403	2.11373	1.96642	1.84801	1.75261	1.67518	1.61159	1.55846	1.51286
	0.0033	2.28906	2.10793	1.96160	1.84420	1.74964	1.67289	1.60986	1.55719	1.51211
	0.0017	2.28310	2.10192	1.95681	1.84042	1.74669	1.67063	1.60815	1.55593	1.51130
	0.0000	2.27567	2.09595	1.95206	1.83668	1.74378	1.66839	1.60645	1.55468	1.51041

5. $a=0.5$

b	t	u_{1j}	u_{2j}	u_{3j}	u_{4j}	u_{5j}	u_{6j}	u_{7j}	u_{8j}	u_{9j}
0.5	0.0067	0.38291	0.52984	0.60956	0.64666	0.63879	0.56091	0.42776	0.24285	0.08468
	0.0050	0.23699	0.33738	0.39813	0.43187	0.43859	0.41163	0.33360	0.20513	0.07574
	0.0033	0.14327	0.21221	0.25805	0.28610	0.29719	0.28916	0.25398	0.17412	0.07008
	0.0017	0.08342	0.13149	0.16597	0.18823	0.19942	0.19931	0.18458	0.14656	0.06848
	0.0000	0.04500	0.08013	0.10585	0.12307	0.13281	0.13555	0.13021	0.11277	0.07452
0.6	0.0067	0.45844	0.63285	0.72486	0.76366	0.74735	0.65991	0.49080	0.27680	0.09615
	0.0050	0.28402	0.40364	0.47463	0.51174	0.51509	0.47800	0.38310	0.23360	0.08583
	0.0033	0.17183	0.25421	0.30830	0.34010	0.35044	0.33716	0.29223	0.19812	0.07922
	0.0017	0.10009	0.15765	0.19861	0.22440	0.23609	0.23348	0.21324	0.16673	0.07714
	0.0000	0.05400	0.09613	0.12685	0.14707	0.15781	0.15955	0.15121	0.12877	0.08352
0.7	0.0067	0.53397	0.73587	0.84015	0.88066	0.85591	0.74991	0.55384	0.31076	0.10763
	0.0050	0.33106	0.46996	0.55113	0.59162	0.59159	0.54438	0.43260	0.26207	0.09593
	0.0033	0.20038	0.29021	0.35855	0.39110	0.40369	0.38516	0.33048	0.22212	0.08836
	0.0017	0.11675	0.18382	0.23130	0.26056	0.27276	0.26764	0.24191	0.18690	0.08581
	0.0000	0.06300	0.11213	0.14785	0.17107	0.18281	0.18355	0.17221	0.14477	0.09252
0.8	0.0067	0.60951	0.83888	0.95544	0.99766	0.96447	0.83991	0.61688	0.34472	0.11910
	0.0050	0.37810	0.53616	0.62763	0.67149	0.66809	0.61075	0.48210	0.29053	0.10602
	0.0033	0.22894	0.33821	0.40880	0.44810	0.45694	0.43316	0.36873	0.24612	0.09750
	0.0017	0.13342	0.20999	0.26397	0.29673	0.30942	0.30181	0.27058	0.20706	0.09448
	0.0000	0.07200	0.12813	0.16885	0.19507	0.20781	0.20755	0.19321	0.16077	0.10152
0.9	0.0067	0.68504	0.94189	1.07074	1.11466	1.07304	0.92991	0.67992	0.37867	0.13057
	0.0050	0.42513	0.60242	0.70413	0.75137	0.74459	0.67713	0.53160	0.31900	0.11611
	0.0033	0.25749	0.38021	0.45905	0.50210	0.51019	0.48116	0.40698	0.27012	0.10663
	0.0017	0.15009	0.23615	0.29664	0.33290	0.34609	0.33598	0.29924	0.22723	0.10314
	0.0000	0.08100	0.14413	0.18985	0.21907	0.23281	0.23155	0.21421	0.17677	0.11052

$a=0.7$

b	t	u_{1j}	u_{2j}	u_{3j}	u_{4j}	u_{5j}	u_{6j}	u_{7j}	u_{8j}	u_{9j}
0.5	0.0067	0.38502	0.53575	0.62280	0.67132	0.67718	0.61788	0.47278	0.27207	0.09561
	0.0050	0.23771	0.33982	0.40438	0.44487	0.46102	0.44353	0.36803	0.23025	0.08585
	0.0033	0.14347	0.21309	0.26077	0.29254	0.30957	0.30883	0.27907	0.19577	0.07983
	0.0017	0.08346	0.13175	0.16702	0.19119	0.20586	0.21070	0.20108	0.16486	0.07854
	0.0000	0.04501	0.08018	0.10619	0.12430	0.13594	0.14177	0.14029	0.12588	0.08633
0.6	0.0067	0.46055	0.63876	0.73810	0.78832	0.78574	0.70788	0.53582	0.30603	0.10708
	0.0050	0.28474	0.40608	0.48088	0.52474	0.53752	0.50990	0.41753	0.25872	0.09594
	0.0033	0.17203	0.25509	0.31102	0.34654	0.36282	0.35683	0.31732	0.21977	0.08897
	0.0017	0.10012	0.15791	0.19960	0.22736	0.24252	0.24487	0.22974	0.18502	0.08720
	0.0000	0.05401	0.09618	0.12719	0.14830	0.16094	0.16577	0.16129	0.14188	0.09533
0.7	0.0067	0.53608	0.74178	0.85339	0.90532	0.89430	0.79788	0.59886	0.33999	0.11855
	0.0050	0.33178	0.47234	0.55738	0.60462	0.61402	0.57628	0.46703	0.28718	0.10604
	0.0033	0.20058	0.29709	0.36127	0.40054	0.41607	0.40483	0.35557	0.24377	0.09811
	0.0017	0.11679	0.18408	0.23236	0.26352	0.27919	0.27903	0.25841	0.20519	0.09587
	0.0000	0.06301	0.11218	0.14819	0.17230	0.18594	0.18977	0.18229	0.15788	0.10435
0.8	0.0067	0.61161	0.84479	0.96868	1.02232	1.00286	0.88788	0.66190	0.37394	0.13003
	0.0050	0.37882	0.53860	0.63388	0.68449	0.69052	0.64265	0.51653	0.31565	0.11613
	0.0033	0.22914	0.33909	0.41152	0.45454	0.46932	0.45283	0.39382	0.26777	0.10725
	0.0017	0.13346	0.21025	0.26502	0.29969	0.31586	0.31320	0.28708	0.22536	0.10454
	0.0000	0.07201	0.12818	0.16919	0.19630	0.21094	0.21377	0.20329	0.17388	0.11333
0.9	0.0067	0.68714	0.94780	1.08398	1.13932	1.11143	0.97788	0.72494	0.40790	0.14150
	0.0050	0.42586	0.60485	0.71038	0.76437	0.76702	0.70903	0.56603	0.34412	0.12622
	0.0033	0.25769	0.38109	0.46177	0.50854	0.52257	0.50083	0.43207	0.29177	0.11639
	0.0017	0.15012	0.23641	0.29769	0.33586	0.35252	0.34737	0.31574	0.24552	0.11320
	0.0000	0.08101	0.14418	0.19019	0.22030	0.23594	0.23777	0.22429	0.18988	0.12233

$a=0.9$

b	t	u_{1j}	u_{2j}	u_{3j}	u_{4j}	u_{5j}	u_{6j}	u_{7j}	u_{8j}	u_{9j}
0.5	0.0067	0.38712	0.54166	0.63604	0.69599	0.71557	0.66584	0.51780	0.30130	0.10654
	0.0050	0.23843	0.34225	0.41064	0.45786	0.48346	0.47543	0.40247	0.25537	0.09596
	0.0033	0.14367	0.21397	0.26349	0.29897	0.32195	0.32849	0.30416	0.21742	0.08959
	0.0017	0.08349	0.13201	0.16808	0.19415	0.21229	0.22209	0.21757	0.18315	0.08859
	0.0000	0.04501	0.08023	0.10653	0.12553	0.13906	0.14799	0.15038	0.15898	0.09814
0.6	0.0067	0.46265	0.64468	0.75134	0.81299	0.82413	0.75584	0.58084	0.33525	0.11801
	0.0050	0.28546	0.40851	0.48714	0.53774	0.55996	0.54180	0.45197	0.28383	0.10605
	0.0033	0.17222	0.25597	0.31374	0.35297	0.37520	0.37649	0.34241	0.24142	0.09873
	0.0017	0.10016	0.15818	0.20075	0.23032	0.24896	0.25626	0.24624	0.20331	0.09726
	0.0000	0.05401	0.09623	0.12753	0.14953	0.16406	0.17199	0.17138	0.15498	0.10714
0.7	0.0067	0.53818	0.74769	0.86663	0.92999	0.93269	0.84584	0.64388	0.36921	0.12948
	0.0050	0.33250	0.47477	0.56364	0.61761	0.63646	0.60818	0.50147	0.31230	0.11615
	0.0033	0.20078	0.29797	0.36399	0.40697	0.42845	0.42449	0.38066	0.26542	0.10786
	0.0017	0.11683	0.18434	0.23341	0.26648	0.28563	0.29043	0.27491	0.22348	0.10593
	0.0000	0.06301	0.11223	0.14853	0.17353	0.18906	0.19599	0.19238	0.17098	0.11614
0.8	0.0067	0.61371	0.85070	0.98192	1.04699	1.04125	0.93584	0.70692	0.40317	0.14095
	0.0050	0.37954	0.54103	0.64014	0.69749	0.71296	0.67455	0.55097	0.34077	0.12624
	0.0033	0.22933	0.33997	0.41424	0.46097	0.48170	0.47249	0.41891	0.28942	0.11700
	0.0017	0.13349	0.21051	0.26608	0.30265	0.32229	0.32459	0.30357	0.24365	0.11459
	0.0000	0.07201	0.12823	0.16953	0.19753	0.21406	0.21999	0.21338	0.18698	0.12514
0.9	0.0067	0.68924	0.95371	1.09721	1.16399	1.14982	1.02584	0.76996	0.43712	0.15243
	0.0050	0.42658	0.60729	0.71664	0.77736	0.78946	0.74093	0.60047	0.36924	0.13633
	0.0033	0.25789	0.38197	0.46449	0.51497	0.53495	0.52049	0.45716	0.31342	0.12614
	0.0017	0.15016	0.23668	0.29875	0.33882	0.35896	0.35876	0.33224	0.29381	0.12326
	0.0000	0.08101	0.14423	0.19053	0.22153	0.23906	0.24399	0.23438	0.20298	0.13414

6.

a	t	u_{1j}	u_{2j}	u_{3j}	u_{4j}	u_{5j}	u_{6j}	u_{7j}	u_{8j}	u_{9j}
0.00	0.0067	0.03242	0.06461	0.10306	0.14174	0.18856	0.23513	0.28933	0.34111	0.39641
	0.0050	0.02994	0.06484	0.09928	0.14129	0.18421	0.23583	0.28605	0.34283	0.39618
	0.0033	0.03073	0.05988	0.09895	0.13868	0.18363	0.22975	0.28803	0.34235	0.39763
	0.0017	0.02125	0.06145	0.09850	0.13645	0.17885	0.23080	0.28065	0.34525	0.40405
	0.0000	0.02210	0.04250	0.10080	0.15450	0.17210	0.20320	0.28950	0.35810	0.40100
0.02	0.0067	0.03492	0.07086	0.11306	0.14549	0.20356	0.23138	0.29933	0.33486	0.39891
	0.0050	0.03494	0.06984	0.10678	0.15629	0.18421	0.25083	0.27855	0.34783	0.39118
	0.0033	0.03073	0.06988	0.10895	0.14368	0.20363	0.22475	0.29803	0.33735	0.39763
	0.0017	0.03125	0.06145	0.10850	0.15645	0.17885	0.25080	0.27065	0.34525	0.39405
	0.0000	0.02210	0.06250	0.10080	0.15450	0.21210	0.20320	0.28950	0.33810	0.40100
0.04	0.0067	0.03742	0.07711	0.12306	0.14924	0.21856	0.22763	0.30933	0.32861	0.40141
	0.0050	0.03994	0.07484	0.11428	0.17129	0.18421	0.26583	0.27105	0.35283	0.38618
	0.0033	0.03073	0.07988	0.11895	0.14868	0.22363	0.21975	0.30803	0.32235	0.39763
	0.0017	0.04125	0.06145	0.11850	0.17645	0.17885	0.27080	0.26065	0.34525	0.38405
	0.0000	0.02210	0.08250	0.10080	0.15450	0.25210	0.20320	0.28950	0.31810	0.40100
0.06	0.0067	0.03992	0.08336	0.13306	0.15299	0.23356	0.22388	0.31933	0.32236	0.40391
	0.0050	0.04494	0.07984	0.12178	0.18629	0.18421	0.28083	0.26355	0.35783	0.38118
	0.0033	0.03073	0.08988	0.12895	0.15368	0.24363	0.21475	0.31803	0.31235	0.39763
	0.0017	0.05125	0.06145	0.12850	0.19645	0.17885	0.29080	0.25065	0.34525	0.37405
	0.0000	0.02210	0.10250	0.10080	0.15450	0.29210	0.20320	0.28950	0.29810	0.40100
0.08	0.0067	0.04242	0.08961	0.14306	0.15674	0.24856	0.22013	0.32933	0.31611	0.40641
	0.0050	0.04994	0.08484	0.12928	0.20129	0.18421	0.29583	0.25605	0.36283	0.37618
	0.0033	0.03073	0.09988	0.13895	0.15868	0.26363	0.20975	0.32803	0.30235	0.39763
	0.0017	0.06125	0.06145	0.13850	0.21645	0.17885	0.31080	0.24065	0.34525	0.36405
	0.0000	0.02210	0.12250	0.10080	0.15450	0.33210	0.20320	0.28950	0.27810	0.40100
1.00	0.0067	0.04492	0.09566	0.15306	0.16049	0.26356	0.21638	0.33933	0.30986	0.40891
	0.0050	0.05494	0.08984	0.13678	0.21629	0.18421	0.31083	0.24855	0.36783	0.37118
	0.0033	0.03073	0.10987	0.14895	0.16368	0.28363	0.20475	0.33803	0.29235	0.39763
	0.0017	0.07125	0.06145	0.14850	0.23645	0.17885	0.33080	0.23065	0.34525	0.35405
	0.0000	0.02210	0.14250	0.10080	0.15450	0.37210	0.20320	0.28950	0.25810	0.40100

10.7

1.

a	t	u_{1j}	u_{2j}	u_{3j}	u_{4j}	u_{5j}	u_{6j}	u_{7j}	u_{8j}	u_{9j}
1.1	0.5	0.10495	0.17858	0.19471	0.13668	0.00050	−0.13813	−0.19538	−0.17880	−0.10498
	0.4	0.20282	0.37577	0.49207	0.53091	0.47983	0.33643	0.17490	0.07496	0.02370
	0.3	0.27083	0.51631	0.71197	0.83522	0.86684	0.79287	0.60677	0.37740	0.17993
	0.2	0.31349	0.60703	0.85946	1.04790	1.14825	1.13718	0.99537	0.71175	0.35370
	0.1	0.33620	0.65664	0.94296	1.17250	1.31824	1.35075	1.24216	0.97167	0.53181
	0.0	0.34315	0.67213	0.96967	1.21330	1.37500	1.42322	1.32705	1.06222	0.61797
1.2	0.5	0.11450	0.19484	0.21245	0.14960	0.00060	−0.15049	−0.21298	−0.19493	−0.11446
	0.4	0.21174	0.39181	0.51186	0.54985	0.49262	0.33768	0.16598	0.06374	0.01638
	0.3	0.27731	0.52876	0.72921	0.85532	0.88693	0.80909	0.61439	0.37729	0.17820
	0.2	0.31702	0.61471	0.87222	1.06630	1.17180	1.16365	1.02040	0.72885	0.36091
	0.1	0.33740	0.66048	0.95179	1.18869	1.34301	1.38310	1.27811	1.00402	0.55065
	0.0	0.34346	0.67448	0.97695	1.22851	1.40000	1.45747	1.36673	1.09991	0.64311
1.3	0.5	0.12406	0.21111	0.23019	0.16163	0.00069	−0.16285	−0.23057	−0.21106	−0.12394
	0.4	0.22065	0.40786	0.53166	0.56879	0.50540	0.33892	0.15707	0.05253	0.00906
	0.3	0.28380	0.54120	0.74646	0.87542	0.90702	0.82531	0.62202	0.37718	0.17646
	0.2	0.32055	0.62240	0.88497	1.08469	1.19534	1.19012	1.04543	0.74595	0.36813
	0.1	0.33860	0.66432	0.96063	1.20488	1.36778	1.41546	1.31405	1.03638	0.56949
	0.0	0.34377	0.67683	0.98423	1.24372	1.42500	1.49172	1.40640	1.13759	0.66825
1.4	0.5	0.13361	0.22737	0.24793	0.17411	0.00079	−0.17521	−0.24816	−0.22719	−0.13342
	0.4	0.22957	0.42390	0.55145	0.58773	0.51818	0.34017	0.14815	0.04131	0.00173
	0.3	0.29029	0.55364	0.76370	0.89553	0.92712	0.84154	0.62964	0.37707	0.17473
	0.2	0.32408	0.63009	0.89772	1.10308	1.21888	1.21659	1.07046	0.76306	0.37534
	0.1	0.33980	0.66815	0.96947	1.22108	1.39255	1.44781	1.35000	1.06873	0.58833
	0.0	0.34408	0.67918	0.99151	1.25894	1.45000	1.52597	1.44607	1.17528	0.69339
1.5	0.5	0.14317	0.24364	0.26568	0.18658	0.00089	−0.18756	−0.26575	−0.24332	−0.14289
	0.4	0.23848	0.43994	0.57124	0.60668	0.53097	0.34142	0.13923	0.03010	−0.00559
	0.3	0.29677	0.56609	0.78094	0.91563	0.94721	0.85776	0.63727	0.37696	0.17299
	0.2	0.32761	0.63777	0.91047	1.12147	1.24243	1.24306	1.09549	0.78016	0.38256
	0.1	0.34100	0.67199	0.97830	1.23727	1.41732	1.48016	1.38595	1.10108	0.60717
	0.0	0.34439	0.68153	0.99879	1.27415	1.47500	1.56022	1.48575	1.21296	0.71853

2.

a	t	u_{1j}	u_{2j}	u_{3j}	u_{4j}	u_{5j}	u_{6j}	u_{7j}	u_{8j}	u_{9j}
0.975	0.5	0.08071	0.02200	0.06760	0.02070	0.02950	−0.01500	0.01590	−0.03120	0.00191
	0.4	0.02391	0.11710	0.05860	0.08210	0.03520	0.06610	0.02140	0.03980	0.04951
	0.3	0.03640	0.06051	0.13160	0.07310	0.11870	0.07160	0.09000	0.10211	0.03790
	0.2	0.03660	0.05090	0.07501	0.16820	0.10949	0.14260	0.15231	0.08810	0.05260
	0.1	0.01450	0.05110	0.08749	0.11140	0.19211	0.19020	0.14070	0.10280	0.05020
	0.0	0.01450	0.05110	0.08749	0.11140	0.19211	0.19020	0.14070	0.10280	0.05020
1.000	0.5	0.07578	0.02462	0.06867	0.01963	0.03057	−0.01607	0.01697	−0.02998	−0.00302
	0.4	0.02160	0.11448	0.06122	0.08317	0.03413	0.06717	0.02262	0.03858	0.04580
	0.3	0.03870	0.05820	0.12898	0.07572	0.11977	0.07282	0.08878	0.09840	0.04160
	0.2	0.03660	0.05320	0.07270	0.16558	0.11442	0.14138	0.14860	0.09180	0.05260
	0.1	0.01450	0.05110	0.08980	0.11140	0.18718	0.19020	0.14440	0.10280	0.05020
	0.0	0.01450	0.05110	0.08980	0.11140	0.18718	0.19020	0.14440	0.10280	0.05020
1.025	0.5	0.07110	0.02700	0.07000	0.01830	0.03190	−0.01740	0.01830	−0.02900	−0.00770
	0.4	0.01930	0.11210	0.06360	0.08450	0.03280	0.06850	0.02360	0.03760	0.04210
	0.3	0.04100	0.05590	0.12660	0.07810	0.12110	0.07380	0.08780	0.09470	0.04530
	0.2	0.03660	0.05550	0.07040	0.16320	0.11910	0.14040	0.14490	0.09550	0.05260
	0.1	0.01450	0.05110	0.09210	0.11140	0.18250	0.19020	0.14810	0.10280	0.05020
	0.0	0.01450	0.05110	0.09210	0.11140	0.18250	0.19020	0.14810	0.10280	0.05020
1.050	0.5	0.06665	0.02915	0.07155	0.01675	0.03345	−0.01895	0.01985	−0.02825	−0.01215
	0.4	0.01700	0.10995	0.06575	0.08605	0.03125	0.07005	0.02435	0.03685	0.03840
	0.3	0.04330	0.05360	0.12445	0.08025	0.12265	0.07455	0.08705	0.09100	0.04900
	0.2	0.03660	0.05780	0.06810	0.16105	0.12355	0.13965	0.14120	0.09920	0.05260
	0.1	0.01450	0.05110	0.09440	0.11140	0.17805	0.19020	0.15180	0.10280	0.05020
	0.0	0.01450	0.05110	0.09440	0.11140	0.17805	0.19020	0.15180	0.10280	0.05020

3.

a	t	u_{1j}	u_{2j}	u_{3j}	u_{4j}	u_{5j}	u_{6j}	u_{7j}	u_{8j}	u_{9j}
1.1	0.5	0.07048	−0.01968	0.11225	0.01115	0.19437	0.11715	0.22202	0.07792	0.20338
	0.4	0.00770	0.07794	0.02317	0.22655	0.12125	0.22802	0.11400	0.21971	0.14020
	0.3	0.00746	0.05055	0.19224	0.13327	0.26020	0.11810	0.22571	0.17628	0.13633
	0.2	0.04285	0.12176	0.16065	0.22589	0.13012	0.25789	0.18038	0.14233	0.12608
	0.1	0.11430	0.15295	0.15541	0.15750	0.22358	0.19240	0.17451	0.16018	0.12600
	0.0	0.11010	0.14795	0.14980	0.15310	0.21978	0.14020	0.17220	0.15818	0.12410
1.2	0.5	0.09046	−0.03966	0.11950	0.01900	0.18724	0.12500	0.22834	0.05794	0.22336
	0.4	0.00770	0.08498	0.01664	0.23380	0.12910	0.23434	0.10840	0.22552	0.14020
	0.3	−0.00548	0.06400	0.19928	0.12674	0.28090	0.11250	0.23152	0.19066	0.12216
	0.2	0.05630	0.10882	0.17410	0.24638	0.11014	0.27808	0.19476	0.12816	0.17046
	0.1	0.11430	0.16640	0.15592	0.15750	0.24356	0.19240	0.17472	0.17457	0.12600
	0.0	0.11010	0.16140	0.14980	0.15310	0.23976	0.14020	0.17220	0.17256	0.12410
1.3	0.5	0.11044	−0.05964	0.12675	0.02685	0.18011	0.13285	0.23466	0.03796	0.24334
	0.4	0.00770	0.09202	0.01011	0.24105	0.13695	0.24066	0.10280	0.23133	0.14020
	0.3	−0.01842	0.07745	0.20632	0.12021	0.30160	0.10690	0.23733	0.20504	0.10799
	0.2	0.06975	0.09588	0.18755	0.26687	0.09016	0.29827	0.20914	0.11399	0.18484
	0.1	0.11430	0.17985	0.15643	0.15750	0.26354	0.19240	0.17493	0.18894	0.12600
	0.0	0.11010	0.17485	0.14980	0.15310	0.25974	0.14020	0.17220	0.18694	0.12410
1.4	0.5	0.13042	−0.07962	0.13400	0.03470	0.17298	0.14070	0.24098	0.01798	0.26332
	0.4	0.00770	0.09906	0.00358	0.24830	0.14480	0.24698	0.09720	0.23714	0.14020
	0.3	−0.03136	0.09090	0.21336	0.11368	0.32230	0.10130	0.24314	0.21942	0.09382
	0.2	0.08320	0.08294	0.20100	0.28736	0.07018	0.31846	0.22352	0.09982	0.19922
	0.1	0.11430	0.19330	0.15694	0.15750	0.28352	0.19240	0.17514	0.20332	0.12600
	0.0	0.11010	0.18830	0.14980	0.15310	0.27972	0.14020	0.17220	0.20132	0.12410
1.5	0.5	0.05040	−0.09960	0.14125	0.04255	0.16585	0.14855	0.24730	−0.00200	0.28330
	0.4	0.00770	0.10610	−0.00295	0.25555	0.15265	0.25330	0.09160	0.24295	0.14020
	0.3	−0.04430	0.10435	0.22040	0.10715	0.34300	0.09570	0.24895	0.23380	0.07965
	0.2	0.09665	0.07000	0.21445	0.30785	0.05020	0.33865	0.23790	0.08565	0.21360
	0.1	0.11430	0.20675	0.15745	0.15750	0.30350	0.19240	0.17535	0.21770	0.12600
	0.0	0.11010	0.20175	0.14980	0.15310	0.29970	0.14020	0.17220	0.21570	0.12410

10.8.

1. $y_0 = 0.921715$; $y_1 = 0.888699$; $y_2 = 0.773750$; $y_3 = 0.546834$; $y_4 = 0.187536$;

2. $y_0 = 0.685597$; $y_1 = 1.010770$; $y_2 = 1.294872$; $y_3 = 1.511112$; $y_4 = 1.637350$; $y_5 = 1.658210$;

3. $y_0 = 0.331090$; $y_1 = 0.058549$; $y_2 = -0.136790$; $y_3 = -0.266859$; $y_4 = -0.346199$;

4. $y_1 = 1.424179$; $y_2 = 1.850109$; $y_3 = 3.147664$; $y_4 = 4.859418$;

10.9.

$y_0 = 2.00000$; $y_1 = 1.47586$; $y_2 = 1.15516$; $y_3 = 1.42747$; $y_4 = 2.07497$; $y_5 = 2.82549$.

10.10

1. (a) $y(x) = (1 + c_1)x + c_2x^3 + c_3x^5$

c_i \ a	0.6	0.8	1.0	1.2
c_1	0.427483	0.664348	1.000000	1.526101
c_2	−0.005122	−0.007058	0	0.036078
c_3	0.000062	0.000155	0	−0.001916

(b) $y(x) = \frac{1}{2\sqrt{x}} + c_1x + c_2x^3 + c_3x^5$

c_i \ a	0.6	0.8	1.0	1.2
c_1	0.855724	1.331451	2.007941	3.072572
c_2	−0.006658	−0.005649	0.016519	0.100953
c_3	0.000052	0.000003	−0.000955	−0.006286

2. (a) $y(x) = \frac{1}{x} + c_1x + c_2x^2 + c_3x^3$

c_i \ a	0.2	0.3	0.4	−0.2	−0.3	−0.4
c_1	0.341729	0.554051	0.807688	−0.269770	−0.386253	−0.493295
c_2	0.019894	0.049588	0.098885	0.014340	0.030148	0.050273
c_3	0.000953	0.003613	0.009735	−0.000652	−0.002027	−0.004446

(b) $y(x) = 1 + (c_1 - 1)x + c_2x^2 + c_3x^3$

c_i \ a	0.2	0.3	0.4	−0.2	−0.3	−0.4
c_1	0.056942	0.092291	0.134476	−0.044954	−0.064352	−0.082168
c_2	0.003204	0.008008	0.016014	0.002278	0.004772	0.007926
c_3	0.000140	0.000539	0.001472	−0.000090	−0.000275	−0.000592

(c) $y(x) = e^{-x} + c_1x + c_2x^2 + c_3x^3$

c_i \ a	0.2	0.3	0.4	−0.2	−0.3	−0.4
c_1	0.097506	0.160772	0.241919	−0.075946	−0.107094	−0.132584
c_2	0.006462	0.016251	0.033250	0.004790	0.009970	0.016080
c_3	0.008053	0.027321	0.065242	−0.007962	−0.026805	−0.063354

3. (a) $y(x) = 1 + (1 + c_1)y + c_2x^2 + c_3x^3$

c_i \ a	0.1	0.2	−0.1	−0.2
c_1	1.234211	1.220339	1.267860	1.287972
c_2	−0.044174	−0.087267	0.045492	0.092565
c_3	0.002587	0.010215	0.002668	0.010869

(b) $y(x) = e^{-x} + c_1x + c_2x^2 + c_3x^3$

c_i \ a	0.1	0.2	−0.1	−0.2
c_1	0.391753	0.387700	0.401505	0.407300
c_2	−0.012803	−0.025290	0.013185	0.026826
c_3	0.000713	0.002813	0.000736	0.003003

(c) $y(x) = \sqrt{x} + c_1x + c_2x^2 + c_3x^3$

c_i \ a	0.1	0.2	−0.1	−0.2
c_1	0.592322	0.585581	0.608689	0.618477
c_2	−0.021485	−0.042445	0.022126	0.045022
c_3	0.001265	0.004995	0.001304	0.005313

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